M. C. GHOSH

ANALYTICAL Statics

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ANALYTICAL STATICS

FOR DEGREE HONOURS AND B. E. STUDENTS
OF INDIAN UNIVERSITIES

WITH NUMEROUS EXAMPLES WORKED

BY

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PREFACE

The book is written in accordance with the syllabus for the B. A. and B. Sc. Honours course of the Calcutta University and other Indian Universities, and is intended to be used as a text-book by the Honours students of those Universities. It is hoped that the post-graduate students and the students of the Engineering Colleges will also find the book very helpful. From my long experience as a teacher of Mathematics and my close contact with students of diverse merits and capacities, I have been convinced that the existing treatises on analytical statics, being too advanced, are unlikely to meet the needs and requirements of the great majority of our students and the need of a suitable text-book for them has been strongly felt. The present treatise is an attempt to meet that need.

In writing the book it has been assumed that the students, for whom it is meant, are acquainted with the elementary principles and theorems of Statics and also possess some knowledge of the usual rules of the Differential and Integral Calculus and also of elementary solid geometry.

Each chapter of the book has been treated elaborately and with due care but without going into unnecessary details and is written in a very simple language easily understandable to the students. I have particularly taken great care in writing the proofs of the theorems to make them as simple, clear and lucid as possible.

To learn Statics it is absolutely necessary for the students to work numerous examples. In order to initiate the students to the correct use of the principles of statics and to the different methods and artifices used in solving various types of problems, a large number of examples, mainly taken from the Honours papers of the Calcutta University, have been solved in details and with full argument. I have no doubt that the solutions will prove to be of immense benefit to the students who will utilise them properly.

I cannot too strongly advise the students to attempt to work out as many as possible of the sums given in the exercises at the end of each chapter. They have been collected from various Examination papers and from the standard treatises on Statics. The examples have been classified and arranged according to their stiffness, for the benefit of the students. Hints for their solutions, where thought necessary, have been given in Answers at the end of the book.

I have consulted the works of most of the well-known authors —especially Dr. Routh, Minchin and Lamb—and my indebtedness to them is gratefully acknowledged. I shall deem my labour to be amply rewarded if the book proves to be of some material help to the students. Any corrections or suggestions for improvement from any reader will be thankfully received.

Lastly I should express my sincere thanks to the publishers who have worked hard and taken the utmost care possible to bring out the book in so short a time.

Calcutta
July, 1969.

M. C. Ghosh

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SYLLABUS

ANALYTICAL STATICS—50 Marks. (Hons.)

Coplanar forces, reduction of a system of forces, equation of the line of action of the resultant. Equilibrium of a particle under constraints (on smooth and rough curves).

Problems of friction : Sliding, tumbling and rolling. Friction on locked and free wheels of a car. Problems involving initial motion of a rod on a plane. Equilibrium of flexible inelastic string under gravity.

Centroids : Plane arcs, plates, bodies of revolution.

Bending moments. Principles of Virtual Work. Energy. Stability of equilibrium.

Forces in three dimensions. Moment of a force about a line. Theorems on moments. Composition of couples. Reduction of a force-system in space. Poinsot's central axis. Invariants of a system of forces. Equation of the central axis. Conditions for reduction to a single force. General conditions of equilibrium, deduction of the general condition from the principle of virtual work.

ANALYTICAL STATICS

INTRODUCTION

The science of Statics treats of forces which acting on a body keep it in equilibrium. It is based on four fundamental principles known as :

I. The Principle of Physical Independence of Forces,
II. The Principle of Transmissibility of Force,
III. The Principle of Action and Reaction,
IV. The Principle of Parallelogram of Forces.

These principles have been derived from various experimental results.

From the first principle is deduced the result that the *effect of a given system of forces remains unaltered if we add to or remove from it any null system of forces.* The second principle states that *a force acting at a point of a rigid body may be taken to act at any other point on its line of action which is rigidly connected with the body.* The third principle states that *to every action there is an equal and opposite reaction.* The fourth principle, the most important one, gives the resultant of two forces acting simultaneously at a point of a rigid body. *If the two forces are represented in magnitude and direction by the two adjacent sides OA, OB of a parallelogram $OACB$ their resultant is represented in magnitude and direction by the diagonal OC.*

To find the conditions of equilibrium of a system of forces acting on a body, the method consists in reducing the system to a single resultant force, if possible and deducing the conditions by making the resultant zero. The resultant of any number of coplanar forces acting at a point may be found by repeated application of the parallelogram of forces or its equivalent the triangle of forces which states that if two forces acting at a point are represented in magnitude and direction by OA and AB, their resultant is represented in magnitude and direction by OB. This leads to the *Polygon of forces*—if any number of forces acting at a point are represented in magnitude and direction by the sides of a *closed* polygon taken in order, the forces are in equilibrium. In the particular case of three forces we have the triangle of forces from which we derive an important and useful result known as *Lami's theorem—if three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.* Its converse is also true. The more useful conditions of equilibrium are derived analytically by the method of resolution and composition of forces. We find $R^2=X^2+Y^2$, $\tan\theta=\frac{Y}{X}$, where R is the magnitude of the resultant, θ its inclination to the axis of x and X, Y the sums of resolved parts of the forces along the axes. This gives the conditions of equilibrium as $X=0$, $Y=0$, *i.e. the sums of the resolved parts of the forces along each of two perpendicular lines should separately vanish.*

Next our problem is to find the resultant of any number of coplanar forces acting on a rigid body. For this we have first to find the resultant of any two parallel forces, like or unlike. This we do by using the first two principles—we apply two equal and opposite forces acting along the same line to the two given forces to get two intersecting forces in their plane and then transfer the points of application of the latter forces to their point of intersection. We derive the following result—*if* P, Q *are two parallel forces either like or unlike and unequal acting at* A *and* B *then their resultant is a force* R *parallel to them and equal to* $P+Q$ *or* $P\backsim Q$ *acting at a point* C *on* AB *which divides* AB *internally or externally in the inverse ratio of* P *and* Q *so that in both the cases, we have* $P.AC=Q.BC$. But when the two parallel forces are equal and unlike, they cannot be reduced to a single force. They, taken together, form a new entity which we call a *Couple*. The effect of a couple acting on a body is to produce rotation and not translation like a force.

To study the properties of couples an important concept is introduced—the *moment of a force about a point* which is defined as the product of the force and the perpendicular distance of its line of action from the point. From experimental results it is found that the moment measures the turning effect of the force about the point. The turning effect of any number of coplanar forces acting at different points of a rigid body is given by an important theorem known as *Varignon's theorem* of moments which states that *the algebraic sum of the moments of any number of coplanar forces about any point in their plane is equal to the moment of their resultant, if any, about the same point.*

We find that in the case of a couple, the sum of the moments of the forces forming the couple about any point in the plane of the couple is constant and equal to the product of either force of the couple and the arm of the couple, which is the perpendicular distance between its forces. This product is called the *moment of the couple* and its sign is positive or negative according as its tends to produce rotation in the counter-clockwise or clockwise direction. Then is deduced the most important property of couples that *any two couples acting in the same plane are equivalent if their moments are equal and of same sign.* This property is also found to hold between couples acting in parallel planes. From this property we easily derive the theorem on composition of couples—*any number of couples acting in the same plane can be compounded into a single couple whose moment is equal to the algebraic sum of the moments of the constituent couples.* We also find that *a couple and a force cannot produce equilibrium* but are together equivalent to a single force which is equal and parallel to the given force with its line of action displaced. Now we are in a position to tackle the problem we set before us—to find the resultant of any number of coplanar forces.

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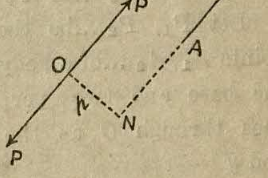
CHAPTER I

COPLANAR FORCES

1·1. We know that two coplanar forces, either intersecting or parallel, can always be reduced to a single force, called their resultant, unless they are two equal and unlike parallel forces, in which case they form a Couple. We also know that any number of coplanar forces acting at a point can be reduced to a single resultant force, the resolved part of which in any direction is equal to the algebraic sum of the resolved parts of the forces; and any number of coplanar couples can be reduced to a single resultant couple whose moment is equal to the algebraic sum of the moments of the constituent couples. Now we shall show that any number of coplanar forces acting at different points of a rigid body can be reduced to a single force and a single couple. First we prove a simple but useful theorem.

Theorem. Any force P acting at a point A of a rigid body can be replaced by an equal and parallel force P acting at an arbitrarily chosen point O of the body together with a couple whose moment is equal to the moment of the given force P about the chosen point O.

At O apply two equal and opposite forces, P, P, each equal and parallel to the force P at A. The resultant of the three forces is the force P at A as the two applied forces balance each other. But the three forces can be combined in a different way. The two equal, unlike parallel forces P at A and P at O form a couple g, whose moment is $P.\ ON$, where ON is the perpendicular from O on P at A, which is the moment of the force P at A about O. And there remains the equal and like parallel force P at O. Thus the force P at A is equivalent to, and therefore can be replaced by, the equal and parallel force P at O and the couple g, whose moment is equal to the moment of P at A about O.



1·2. Reduction of a given system of coplanar forces.

Theorem. Any system of coplanar forces acting on a rigid body can be reduced to a single resultant force acting at an arbitrarily chosen point O and a single resultant couple whose moment is equal to the algebraic sum of the moments of the given forces about O.

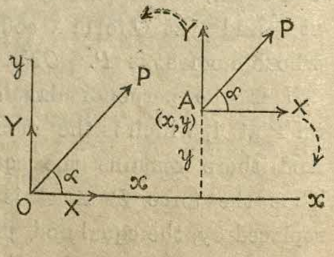
Let P_1, P_2, ... be the given forces acting at the points A_1, A_2, ...of the body. Replace P_1 at A_1 by an equal and parallel force P_1 at O and a couple g_1 whose moment is equal to the moment of P_1 at A_1 about O. Replace P_2 at A_2 by an equal and parallel force P_2 at O and a couple g_2 whose moment is equal to the moment of P_2 at A_2 about O. Similarly replace all the other given forces. Thus the whole system of forces will be replaced by forces P_1, P_2, ... all acting at O and couples g_1, g_2, ... The forces P_1, P_2, ... acting at O can be reduced to a single resultant force R at O and the couples g_1, g_2, ... can be reduced to a single resultant couple G, whose moment is equal to the sum of the moments of the couples, which again, is equal to the sum of the moments of the given forces P_1, P_2, ... about O. Thus the given system is reduced to a single force R at O and a single couple G.

The arbitrarily chosen point O, with respect to which the system is reduced, is called the *base*. Since R is the resultant of the given forces moved parallel to themselves to act at the chosen base, it follows that R is the same for all bases but G being equal to the algebraic sum of the moments of the forces about the chosen base is generally different for different bases.

1·3. Analytical treatment of reduction.

Let P_1, P_2,...be the given coplanar forces acting at the points A_1, A_2,...of a body. Choose any point O in the plane as the base and two perpendicular lines through O as the axes of x and y.

Let (x_1, y_1), (x_2, y_2),...be the co-ordinates of A_1, A_2,......and let the components of P_1, P_2,...... parallel to the axes be (X_1, Y_1), (X_2, Y_2).......



Now, force P_1 at $A_1 \equiv$ force X_1 at A_1 and force Y_1 at A_1

And force X_1 at $A_1 \equiv$ a force X_1 at O along Ox and a couple of moment $-y_1X_1$ (moment of X_1 at A_1 about O; counter-clockwise rotation is considered positive).

And force Y_1 at $A_1 \equiv$ a force Y_1 at O along Oy and a couple of moment $+x_1Y_1$. (moment of Y_1 at A_1 about O)

The two couples can be compounded into a single couple g_1 of moment $(x_1Y_1-y_1X_1)$.

Thus the force P_1 at A_1 is equivalent to a force X_1 at O along Ox, a force Y_1 at O along Oy and a couple g_1 of moment $(x_1Y_1-y_1X_1)$. Similarly for the other forces.

Hence the given system is equivalent to

(*a*) forces $X_1, X_2, \ldots$ at O along Ox,

(*b*) forces $Y_1, Y_2, \ldots$ at O along Oy,

and (*c*) couples $g_1, g_2, \ldots$ of moments $(x_1Y_1-y_1X_1)$, $(x_2Y_2-y_2X_2), \ldots\ldots$

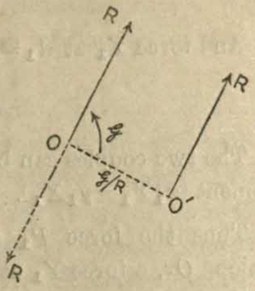
The forces along Ox can be compounded into a single force $X=X_1+X_2+\ldots$; the forces along Oy can be compounded into a single force $Y=Y_1+Y_2+\cdots$; and the couples $g_1, g_2, \ldots$ can be compounded into a single couple G of moment $\Sigma(x_1Y_1-y_1X_1)$.

The given system has been reduced to a single force X along Ox, a single force Y along Oy and a couple G, where X=sum of the resolved parts of the given forces in the direction Ox, Y=sum of the resolved parts of the given forces in the direction Oy and moment of G=sum of the moments of the given forces about O.

The forces X, Y at O can be compounded into a single force $R=\sqrt{X^2+Y^2}$ acting at O and making with Ox an angle θ where $\tan\,\theta=\dfrac{Y}{X}$.

Thus the given system reduces to a single force R at O and a single couple G.

1·4. We have seen that a system of coplanar forces can be reduced to a single force R at any base O and a single couple G. Now we shall show that they can further be reduced to either a single force or a single couple. If $R=0$, the system reduces to a single couple G. If $R\neq 0$ and $G=0$, the system reduces to a single force R at O. If $R \neq 0$, $G \neq 0$, the system reduces to a single force equal and parallel to R but with the line of action displaced. For, at O draw $OO' = \frac{G}{R}$ perpendicular to R and at O, O' take two opposite forces each equal and parallel to R, the force at O balancing the resultant force R at O. Then couple G can be replaced by couple $\left(R, \frac{G}{R}\right)$ and the system reduces to the single force R at O'. Thus when $R \neq 0$, the system always reduces to a single force. When $R = 0$ and $G = 0$, the system, of course, is in equilibrium. Thus *unless the system is in equilibrium it can be reduced either to a single force or to a single couple.*

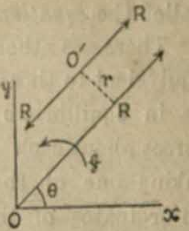


1·5. Suppose a given system has been reduced w.r. to a base O to a force X along Ox, a force Y along Oy and a couple $G = \Sigma(x_1Y_1 - y_1X_1)$. If any other point A (x, y) be chosen as base, the corresponding couple G' will be equal to the algebraic sum of the moments of the forces about A and by transferring the origin to A, we find,

$$
\begin{aligned}
G' &= \Sigma[(x_1 - x)Y_1 - (y_1 - y)X_1] \\
&= \Sigma(x_1Y_1 - y_1X_1) - x\Sigma Y_1 + y\Sigma X_1 \\
&= G - xY + yX. \qquad \text{(For fig. see art. 1·9)}
\end{aligned}
$$

This gives an expression for the algebraic sum of the moments of the given forces about any point (x, y) when the system has been reduced with the origin as base. It may be observed that if the given system is equivalent to a couple, then $X = 0$, $Y = 0$ and G' becomes equal to G. Hence *when the system is equivalent to a couple, the algebraic sum of the moments of the forces about every point in the plane is the same and equal to the moment of the couple.*

1·6. When the system has been reduced w.r. to the base O to a force R at O and a couple G, we can easily find the reduction w.r. to another base O' thus: Replace R at O by an equal and parallel force R at O' and a couple of moment Rr, where r is the perpendicular distance of O' from R at O. The two couples compound to a single couple $G'=G+Rr$ and the reduction w.r. to base O' gives a force R at O' and a couple G', whose moment is equal to sum of moment of couple G and moment of force R at O about O'. This proves again that R is independent of the base chosen but G depends on it and that if $R=0$ i.e. the system is equivalent to a couple, G has the same value for all bases.



1·7. When the system reduces to a single resultant force R, the equation of its line of action can be easily found. Take any point O as base and rect. axes Ox, Oy. Let the system reduce to forces X, Y along the axes and a couple G, and let (x', y') be any point on the line of action of R. Then algebraic sum of the moments of the forces about (x', y') is zero.

$$\therefore\quad G' \equiv G - x'Y + y'X = 0. \quad \text{(Art. 1·5)}$$

$\therefore$ *The equation of the line of action of* R, which is the locus of (x', y'), *is* $G - xY + yX = 0$.

1·8 Conditions of equilibrium of a system of coplanar forces.

Take any point O in the plane of the forces as base and any two perpendicular lines at O and lying in the plane as axes of x, y. Let the system reduce to a single resultant force R at O and a couple G. Since a force and a couple cannot balance each other, the system will be in equilibrium if and only if $R=0$, $G=0$. Since $R^2=X^2+Y^2$, $R=0$ if and only if ; $X=0$, $Y=0$. Hence the system will be in equilibrium if and only if $X=0$, $Y=0$, $G=0$. Now X and Y are the sums of the resolved parts of the forces in the directions of the axes Ox, Oy respectively and G is the algebraic sum of the moments of the forces about O and the base O and the axes are chosen arbitrarily ; it follows, therefore that the *necessary and sufficient* conditions for the system to be in equilibrium are (1) *The algebraic sums of the resolved parts of the forces in two perpendicular directions in the plane should be separately zero* and (2) *the algebraic sum of the moments of the forces about a point in the plane should be zero.*

The conditions in (1) lead to two equations called the *equations of resolution* and the condition in (2) leads to another equation called the *equation of moments*.

There are other forms of conditions of equilibrium which are equivalent to those given above. One such is : The system will be in equilibrium if the algebraic sum of the moments of the forces about each of three non-collinear points be zero. For taking one of the points as origin and (x_1, y_1), (x_2, y_2) as the co-ordinates of the other two, we get, $G=0$, $G-x_1Y+y_1X=0$, $G-x_2Y+y_2X=0$, which give $X=0$, $Y=0$, $G=0$.

1·9. Astatic equilibrium, Astatic centre :

A given system of coplanar forces acting at given points of a body are in equilibrium. Now let each force be turned about its point of application in the plane of the forces through the same angle in the same sense. If the system be still in equilibrium whatever the common angle of rotation may be, then the system is said to be in *Astatic equilibrium*.

Take any point O in the plane of the forces as base and choose rect. axes Ox, Oy. Let $P_1, P_2, \ldots$ be the forces acting at points $A_1, A_2, \ldots$ whose co-ordinates are $(x_1, y_1), (x_2, y_2), \ldots$ and let $(X_1, Y_1), (X_2, Y_2), \ldots$ be the components of $P_1, P_2, \ldots$ in the directions of the axes and let $P_1, P_2, \ldots$ make angles $\alpha_1, \alpha_2, \ldots$ with Ox.

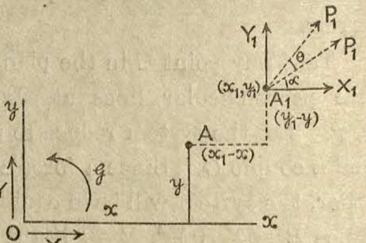
Then the system reduces to forces X, Y along Ox, Oy and a couple G, where

$$X=\Sigma X_1=\Sigma P_1 \cos\alpha_1 \ldots (1)$$

$$Y=\Sigma Y_1=\Sigma P_1 \sin\alpha_1 \ldots (2)$$

and $G=\Sigma(x_1Y_1-y_1X_1)$

$$=\Sigma P_1(x_1 \sin\alpha_1-y_1 \cos\alpha_1) \ldots\ldots (3)$$



Now let each force be rotated counter-clockwise about its point of application through an angle θ. Then $P_1, P_2, \ldots$ make with Ox angles $\alpha_1+\theta, \alpha_2+\theta, \ldots$. If the new system of forces reduce to X', Y', G', we have,

$$X'=\Sigma P_1 \cos(\alpha_1+\theta)=\cos\theta\,\Sigma P_1 \cos\alpha_1-\sin\theta\,\Sigma P_1 \sin\alpha_1$$
$$=X\cos\theta-Y\sin\theta \ldots (4)$$

$$Y'=\Sigma P_1 \sin(\alpha_1+\theta)=\cos\theta\,\Sigma P_1 \sin\alpha_1+\sin\theta\,\Sigma P_1 \cos\alpha_1$$
$$=Y\cos\theta+X\sin\theta \ldots (5)$$

$$\begin{aligned}G' &= \Sigma P_1[x_1 \sin(\alpha_1+\theta) - y_1 \cos(\alpha_1+\theta)] \\ &= \Sigma P_1[\cos\theta(x_1 \sin\alpha_1 - y_1\cos\alpha_1) + \sin\theta\,(x_1\cos\alpha_1 + y_1\sin\alpha_1)] \\ &= \cos\theta\,\Sigma P_1(x_1\sin\alpha_1 - y_1\cos\alpha_1) + \sin\theta\,\Sigma P_1(x_1\cos\alpha_1 + y_1\sin\alpha_1) \\ &= G\cos\theta + V\sin\theta \quad \ldots \ldots \ldots \quad (6)\end{aligned}$$

where $V = \Sigma(x_1 P_1 \cos\alpha_1 + y_1 P_1 \sin\alpha_1)$

$= \Sigma(x_1 X_1 + y_1 Y_1)$, called the Virial of the forces.

If the given system of forces be in equilibrium, then $X=0$, $Y=0$, $G=0$ and $\therefore$ $X'=0$, $Y'=0$, $G'=V\sin\theta$. Thus the system, after rotation, becomes equivalent to a couple of moment $V\sin\theta$. Hence *the system will be in Astatic equilibrium if and only if* $V=0$. When the given system is equivalent to a couple we have $X=0$, $Y=0$, $G\neq 0$. $\therefore$ $X'=0$, $Y'=0$, $G'=G\cos\theta+V\sin\theta$. Thus the system after rotation, is equivalent to a couple of moment $G\cos\theta+V\sin\theta$. If the given forces have a single resultant, then X, Y both cannot vanish and $\therefore$ X', Y' cannot both be zero and so after rotation the system will have a single resultant, whose equation is $G'-xY'+yX'=0$ *or* $\cos\theta(G-xY+yX)+\sin\theta(V-xX-yY)=0$. Whatever value θ may have, the resultant always passes through the point of intersection of the lines $G-xY+yX=0$, $V-xX-yY=0$, whose co-ordinates are $\dfrac{VX+GY}{X^2+Y^2}$, $\dfrac{VY-GX}{X^2+Y^2}$.

Hence we have the theorem :—*If a system of coplanar forces acting at different points of a body have a single resultant and if each force be turned in the plane of the forces about its point of application through the same angle in the same sense, their resultant will always pass through a fixed point.*

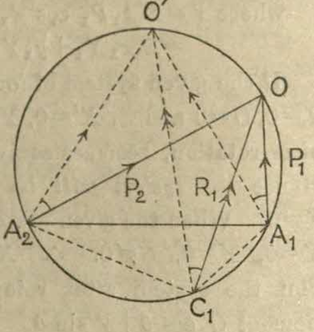
This point is called the *Astatic centre* or simply the *centre of the forces.*

Now we shall show that *the resultant also turns about the astatic centre through the same angle* θ. Let ϕ, ϕ' be the angles the resultant makes with Ox before and after the forces are rotated. Then $\tan\phi=\dfrac{Y}{X}$ and $\tan\phi'=\dfrac{Y'}{X'}$

$$\therefore \quad \tan\phi' = \frac{X\sin\theta+Y\cos\theta}{X\cos\theta-Y\sin\theta} = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \tan(\theta+\phi)$$

$\therefore$ $\phi'=\phi+\theta$. *i.e.* the resultant has rotated through the same angle θ in the same sense.

1·10. Let P_1, P_2 be two forces acting at points A_1, A_2 and let them meet at a point O. If P_1, P_2 be turned about A_1, A_2 through same angle in same sense in the plane OA_1A_2, the angle between P_1, P_2 will remain unaltered and so they will always meet on the circle OA_1A_2. Again, the resultant R_1 of P_1, P_2 will remain unaltered in magnitude and will always make the same angles with P_1 and P_2. Hence the resultant will always pass through the same point C_1 on the circle OA_1A_2. Hence *if two coplanar forces be rotated about their points of application through the same angle in the same sense, their resultant always passes through a fixed point.* If we now take a third force P_3 and rotate the forces P_1, P_2, P_3 about their points of application through same angle, the resultant R_1 of P_1, P_2 will rotate round C_1 through same angle and, therefore, the resultant of R_1 and P_3 *i.e.* the resultant of P_1, P_2, P_3 will always pass through a fixed point. Proceeding thus we can establish the same result for a system of coplanar forces ; this has been proved analytically in previous article. It also follows that when the forces are turned round their points of application through an angle $\sphericalangle$, their resultant also turns round by the *same angle* $\sphericalangle$ about the astatic centre.



Examples Worked

Ex. 1. Prove that in all cases a system of coplanar forces can be replaced by two forces, one of which acts through a given point and the other along a given straight line. [C. H. 1965, '67]

Let A be the given point and BC the given line. The system can be reduced to a force R acting at A and a couple G. The couple can be replaced by two equal unlike parallel forces S, S, of suitable magnitude, one of which acts along BC and the other at A. The two forces R and S at A can be compounded into a single force T at A. Thus the system has been replaced by a force T acting at A and a force S acting along BC.

Ex. 2. The moment for one system of coplanar forces about each of three non-collinear points in the same plane as the forces is equal to the moment of another system about the same points. Prove that the two systems are equivalent. Hence show that any system of coplanar forces may be replaced by three forces acting round the sides of a given triangle ABC in the plane of the forces. Also find the forces. [Allahabad 1930]

Let the first system be denoted by S_1 and the second by S_2 and let S be the system of forces formed by adding to the forces of S_1, the forces of S_2 each reversed in direction. Then evidently the system S has zero moments about each of the three non-collinear points and therefore the system is in equilibrium. Thus S_1 is balanced by S_2 reversed and so they are equivalent. For the 2nd. part, let L, M, N be the moments of the given system about the vertices A, B, C of the $\triangle ABC$, whose sides are a, b, c and $\triangle$ the area. Forces P, Q, R acting along BC, CA, AB will be equivalent to the given system, if the sum of their moments about A, B, C be L, M and N respectively. For this we must have, $\frac{P}{BC}\cdot 2\triangle =$ sum of moments of P, Q, R about $A = L$

or, $P = \frac{aL}{2\triangle}$. Similarly $Q = \frac{bM}{2\triangle}$, $R = \frac{cN}{2\triangle}$.

Thus the forces $\frac{aL}{2\triangle}$ along BC, $\frac{bM}{2\triangle}$ along CA and $\frac{cN}{2\triangle}$ along AB are equivalent to the given system.

Ex. 3. A system of forces in one plane acts on a rigid body and the moments of the system about three non-collinear points in the plane are α, β, γ. Prove that

(*i*) if α, β, γ are not all equal, the system is equivalent to a single force ;

(*ii*) if $\alpha = \beta = \gamma \neq 0$, the system is equivalent to a couple ;

(*iii*) if $\alpha = \beta = \gamma = 0$, the system is in equilibrium.

[C. H. 1960, '68]

Take any rect axes Ox, Oy in the plane of the forces and let the system be reduced w.r. to base O to the forces X, Y along Ox, Oy and the couple G. Let the co-ordinates of the three non-collinear points be (x_1, y_1), (x_2, y_2), (x_3, y_3). Considering moments of the system about these points, we have

$$G-x_1Y+y_1X=\alpha \cdots\cdots(1)$$
$$G-x_2Y+y_2X=\beta \cdots\cdots(2)$$
$$G-x_3Y+y_3X=\gamma \cdots\cdots(3).$$

These eqns. determine X, Y and G. Since the determinant of the co-efficients,

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2\triangle \neq 0,$$

as the given points are non-collinear, the eqns. give unique values of X, Y, G, which by Cramer's Rule,

are $X=\begin{vmatrix} 1 & x_1 & \alpha \\ 1 & x_2 & \beta \\ 1 & x_3 & \gamma \end{vmatrix} \div 2\triangle$, $-Y=\begin{vmatrix} 1 & \alpha & y_1 \\ 1 & \beta & y_2 \\ 1 & \gamma & y_3 \end{vmatrix} \div 2\triangle$,

$G=\begin{vmatrix} \alpha & x_1 & y_1 \\ \beta & x_2 & y_2 \\ \gamma & x_3 & y_3 \end{vmatrix} \div 2\triangle$.

From these values, the results easily follow :

(*i*) When α, β, γ are not all equal, X and Y cannot both vanish, for then from equs. (1), (2), (3) above, we shall have $G=\alpha=\beta=\gamma$. As X, Y are not both zero, R is not zero. ∴ the system reduces to a single force, whether G is zero or not.

(*ii*) When $\alpha=\beta=\gamma\neq 0$, the values of X, Y become both zero while $G\neq 0$; ∴ the system reduces to a couple.

(*iii*) When $\alpha=\beta=\gamma=0$, we have $X=0$, $Y=0$, $G=0$

∴ the system is in equilibrium.

Observe that if the system reduces to a *single force at O*, then $G=0$ *i.e.* $\begin{vmatrix} \alpha & x_1 & y_1 \\ \beta & x_2 & y_2 \\ \gamma & x_3 & y_3 \end{vmatrix}=0$ or $\Sigma\alpha(x_2y_3-x_3y_2)=0$.

Ex. 4. The algebraic sums of the moments of a system of coplanar forces about points whose co-ordinates are (1, 0) (0, 2) and (2, 3) referred to rectangular axes are H, $2H$ and $3H$ respectively. Find the tangent of the angle which the direction of the resultant force makes with the axis of x. [C. H. 1968]

Let the given system reduce to the forces X, Y along the axes of x, y and the couple G. Considering moments of the system about given points, we get, $G-Y=H$, $G+2X=2H$. $G-2Y+3X=3H$. Solving, we get, $X=\frac{3}{5}H$, $Y=-\frac{1}{5}H$, $G=\frac{4}{5}H$, Since X, Y are not zeros, the system reduces to a single resultant force inclined at an angle θ to the x axis where $\tan\theta=\dfrac{Y}{X}=-\dfrac{1}{3}$.

The equation of the line of action of the resultant is

$$\tfrac{4}{5}H-x(-\tfrac{1}{5}H)+y.\tfrac{3}{5}H=0 \quad or \quad x+3y+4=0.$$

Ex. 5. A system of co-planar forces is such that the sum of their moments about any point on a given straight line in the plane is always the same. Show that if the resultant is not zero, it must be parallel to the given line. [C. H. 19 4]

Take the given line as the x axis and choose rect. axes. Let the system be reduced to the forces X, Y along Ox, Oy and the couple G. Any point on the x-axis is $(x', 0)$ and the moment of the system about it is constant, say k, for all x'. $\therefore$ $k=G-x'Y$ or $(G-k)-x'Y=0$, for all values of x'. $\therefore$ $Y=0$ and $G=k$. Since resultant is not zero, $X\neq 0$. $\therefore$ Resultant makes with x axis an angle θ where $\tan\theta=\dfrac{Y}{X}=0$ or $\theta=0$ i.e. the resultant is parallel to the x-axis which is the given line. Observe that if X be zero, then $R=0$ and system is equivalent to the couple $G=k$.

Ex. 6. Moments of the resultant R of a system of coplanar forces about three points O, A and B lying in the plane of the forces are G, $G+J_1$ and $G+J_2$. If referred to O as origin the polar co-ordinates of A and B are (r_1, θ_1) and (r_2, θ_2), show that

$$R^2\sin^2(\theta_1-\theta_2)=\frac{J_1^2}{r_1^2}+\frac{J_2^2}{r_2^2}-2\cdot\frac{J_1J_2}{r_1r_2}\cos(\theta_1-\theta_2). \quad \text{[C.H. 1954]}$$

The cartesian co-ordinates of A, B are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$. Let the system reduce to forces X, Y along the x and y axes and the couple G. Considering moments of the system about A and B, we get

$$G+J_1=G-r_1\cos\theta_1 Y+r_1\sin\theta_1 X \quad\text{or}\quad \frac{J_1}{r_1}=\sin\theta_1 X-\cos\theta_1 Y,$$

$$G+J_2=G-r_2\cos\theta_2 Y+r_2\sin\theta_2 X \quad\text{or}\quad \frac{J_2}{r_2}=\sin\theta_2 X-\cos\theta_2 Y.$$

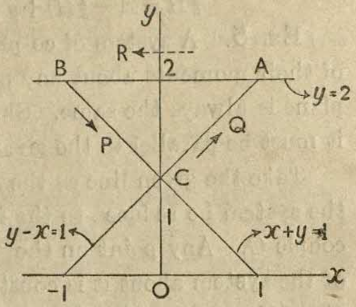
Solving for X, Y we get,

$$X\sin(\theta_1-\theta_2)=\frac{J_1}{r_1}\cos\theta_2-\frac{J_2}{r_2}\cos\theta_1$$

$$Y\sin(\theta_1-\theta_2)=\frac{J_1}{r_1}\sin\theta_2-\frac{J_2}{r_2}\sin\theta_1$$

Since $X^2+Y^2=R^2$, by squaring and adding the result follows.

Ex. 7. Three forces P, Q, R act along the sides of the triangle, formed by the lines $x+y=1$, $y-x=1$, $y=2$. Find the equation of the line of action of the resultant. [I]



Let Ox, Oy be the axes of co-ordinates and let ABC be the triangle formed by the given lines, the sides BC, CA, AB being along the lines $x+y=1$, $y-x=1$, and $y=2$ respectively so that the forces P, Q, R act along the sides BC, CA and AB. $\angle ABC=\angle BAC=45°$.

Now X= sum of resolved parts of the forces along x axis

$$=P\cos 45°+Q\cos 45°-R=\frac{P+Q}{\sqrt{2}}-R$$

Y= sum of resolved parts of the forces along y axis

$$=Q\sin 45°-P\sin 45°=\frac{Q-P}{\sqrt{2}}$$

G= alg. sum of moments of the forces about O

$$=-P.1.\sin 45°-Q\,1.\sin 45°+R.2=2R-\frac{P+Q}{\sqrt{2}}$$

Equation of line of action of the resultant is $G-xY+yX=0$

or $2R-\dfrac{P+Q}{\sqrt{2}}-x.\dfrac{Q-P}{\sqrt{2}}+y\left(\dfrac{P+Q}{\sqrt{2}}-R\right)=0$

or $P(x+y-1)+Q(y-x-1)-\sqrt{2}.R(y-2)=0$.

Also magnitude of resultant

$$=\sqrt{X^2+Y^2}=\sqrt{P^2+Q^2+R^2-\sqrt{2}.R(P+Q)}.$$

Ex. 8. A plane system of forces is equivalent to a couple of moment M, and if the forces are turned through a right angle, about their respective points of application in the same sense, they are equivalent to a couple N. Prove that when each force is turned about its point of application through an angle α in the same sense, the system will be in equilibrium if $\tan\alpha=-\dfrac{M}{N}$.

[C. H. 1959]

Show also that if the forces be turned through the angle $2\tan^{-1}\left(\frac{N}{M}\right)$, they would still be equivalent to a couple of moment M. [Nagpur 1948]

In the plane of the forces let any rect. axes Ox, Oy be chosen. Let X and Y be the sums of the resolved parts of the forces along Ox and Oy respectively and let G be the sum of the moments of the forces about O. Since the system is equivalent to a couple of moment M, we have $X=0$, $Y=0$, $G=M$. When the forces have been turned through an angle θ, let X', Y', G' be the corresponding values of X, Y, G.

We know, $X'=X\cos\theta-Y\sin\theta$,

$Y'=X\sin\theta+Y\cos\theta$,

$G'=G\cos\theta+V\sin\theta$; where V is the Virial of the forces.

$\therefore\ X'=0,\ Y'=0,\ G'=M\cos\theta+V\sin\theta.$

i.e. system becomes a couple of moment $M\cos\theta+V\sin\theta$.

When $\theta=90^\circ$, the couple is given to be N.

$\therefore\ N=M\cos 90^\circ+V\sin 90^\circ=V$

$\therefore\ G'=M\cos\theta+N\sin\theta.$

If system is in equilibrium when $\theta=\alpha$, we must have

$$G'\equiv M\cos\alpha+N\sin\alpha=0 \quad \therefore\ \tan\alpha=-\frac{M}{N}.$$

Suppose when $\theta=\beta$, the system remains equivalent to couple M. $\therefore\ G'\equiv M\cos\beta+N\sin\beta=M$

or, $N\sin\beta=(1-\cos\beta)M$,

$$or,\ \frac{1-\cos\beta}{\sin\beta}=\frac{N}{M}\ or,\ \tan\frac{\beta}{2}=\frac{N}{M} \quad \therefore\ \beta=2\tan^{-1}\left(\frac{N}{M}\right).$$

Ex. 9. When a system of coplanar forces in equilibrium continues in equilibrium for all displacements in the plane of the forces, show that the astatic centre of any number of them must be coincident with that of the remainder.

Let the system be divided into any two groups S_1 and S_2; and let the resultants of the two groups be R_1 and R_2 and their astatic centres C_1 and C_2 respectively. R_1 always acts at C_1 and R_2 at C_2 through whatever common angle the forces are rotated and since the system is always in equilibrium, R_1 always balances R_2 and therefore R_1, R_2 always act along same straight line. $\therefore$ R_1 always passes through C_2. Hence R_1 always passes through C_1 and C_2. $\therefore$ Unless C_1 and C_2 coincide, R_1 will always act along the fixed line C_1C_2. But this is not true as when the forces are rotated through an angle θ, R_1 also rotates about C_1 through the same angle θ. Hence C_1 and C_2 must be coincident points *i.e.* both the groups S_1 and S_2 have the same astatic centre.

Ex. 10. Show that as the forces are rotated, the value of $\frac{G}{V}$ at any assumed base O is always equal to the tangent of the angle which the straight line joining O to the centre C of the forces makes with the direction of the resultant force R, while the value of G^2+V^2 is invariable and equal to $R^2 . OC^2$.

Referred to rect. axes at O, the co-ordinates of C are

$$\xi=\frac{VX+GY}{X^2+Y^2},\quad \eta=\frac{VY-GX}{X^2+Y^2}$$

$$\therefore\quad OC^2=\xi^2+\eta^2=\frac{(V^2+G^2)(X^2+Y^2)}{(X^2+Y^2)^2}=\frac{V^2+G^2}{X^2+Y^2}=\frac{V^2+G^2}{R^2}$$

$\therefore$ $R^2 . OC^2=V^2+G^2$. As R is always the same and O, C are fixed points, $\therefore$ V^2+G^2 is invariable. If R and OC make angles ϕ and ψ with Ox, we have

$$\tan\phi=\frac{Y}{X} \text{ and } \tan\psi=\frac{\eta}{\xi}=\frac{VY-GX}{VX+GY}$$

$$=\frac{\frac{Y}{X}-\frac{G}{V}}{1+\frac{Y}{X}\cdot\frac{G}{V}}=\frac{\tan\phi-\tan\beta}{1+\tan\phi\tan\beta},\text{ where } \frac{G}{V}=\tan\beta.$$

or, $\tan\psi=\tan(\phi-\beta)$ $\therefore$ $\psi=\phi-\beta$ *or,* $\beta=\phi-\psi=$ angle between R and OC. Thus $\frac{G}{V}=$ tan. of angle between R and OC.

There is another a very simple solution. We know that G is the moment of the system about origin O before the forces are rotated and V is the moment of the system after the forces are rotated through 90°. But the moment of the system about O is the moment of the resultant force R about O and when the forces are rotated through 90°, R also rotates about C through 90°. Therefore if β be the angle R makes with OC, we have, $G=R.\ OC\sin\beta$ and $V=R.\ OC\cos\beta$, whence the results follow.

EXAMPLES

1. Prove that a system of coplanar forces can always be reduced to (1) two forces acting at two given points, (2) three forces acting along the sides of a given triangle, in same plane.

Hint. (1) Reduce taking one point as base and replace couple by two forces acting at given points (2) see Ex. 1. worked. Replace T by components along sides.

2. If the sums of the moments round three points not in one straight line are equal, the forces are either in equilibrium or equivalent to a couple. [C. H. 1965]

Hint. See Ex. 3. worked.

3. If each of a system of coplanar forces be replaced by three forces acting along the sides of a triangle in the plane of the forces, of type $p.\,BC$, $q.\,CA$, and $r.\,AB$, show that the necessary and sufficient conditions that the system reduces to a couple are that $\Sigma p = \Sigma q = \Sigma r$. [London]

Hint. See Ex. 2. worked.

4. The moments of a system of forces about the points (0, 0), $(a, 0)$, $(0, a)$ are aW, $2aW$, $3aW$ respectively. Find the components of their resultant parallel to the co-ordinate axes (which are rectangular) and the equation to its line of action. [C. H. 1960, 1968]

Hint. See Ex. 4. worked.

5. A system of coplanar forces has the total moments H, $2H$, and $3H$ about the points (0, 0), (0, 1), and (2, 4) respectively. Find the magnitude and the equation of the line of action of the resultant. [C. H. 1966]

6. A system of coplanar forces has the total moments H, $2H$ respectively about points whose co-ordinates are $(2a, 0)$ $(0, a)$ referred to fixed rectangular axes. The total resolved parts of the forces along the line $y=x$ vanishes. Find the points in which the line of action of the resultant meets the co-ordinate axes. [C. H. 1961]

7. M_1, M_2, M_3 are the moments of a system of forces, acting in the xy-plane, about three non-collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. If the resultant of the system is a single force at the origin, show that $M_1(x_2y_3 - x_3y_2) + M_2(x_3y_1 - x_1y_3) + M_3(x_1y_2 - x_2y_1) = 0$. Find the direction and magnitude of the resultant. [C. H. 1963]

Hint. See Ex. 3. worked.

8. The algebraic sums of the moments of a system of forces about points whose co-ordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) referred to Ox and Oy as axes are G_1, G_2, G_3 and G_4 respectively. Show that

$$\begin{vmatrix} 1 & x_1 & y_1 & G_1 \\ 1 & x_2 & y_2 & G_2 \\ 1 & x_3 & y_3 & G_3 \\ 1 & x_4 & y_4 & G_4 \end{vmatrix} = 0$$

[Bombay, 1954]

9. The straight line $4x + 3y = 5$ meets the rectangular axes Ox and Oy at the points A and B respectively. If forces P, Q, R act along the lines

OB, OA and AB, find the magnitude of the resultant and the equation of the line of action. [C. H. 1958]

Hint. Proceed as in Ex. 7. worked.

10. Three forces P, Q, R act along the sides of a triangle formed by the lines $x+y=3$, $2x+y=1$, and $x-y=-1$. Find the equation of the line of action of the resultant.

Hint. Proceed as in Ex. 7. worked.

11. The couple components of a system of coplanar forces when reduced with respect to two different bases O and O' are G and G' respectively. Show that the couple component when the system is reduced with respect to the middle point of OO' is $\frac{1}{2}(G+G')$.

Hint. Couple for O' = couple for O + moment of R at O about O'. See Art. 1·6.

12. A system of coplanar forces reduces to a couple whose moment is G and when each force is turned round its point of application in the plane of the forces through a right angle, the system reduces to a couple of moment $G\sqrt{3}$. Prove that when each force is turned through 60° in the same plane, the system is also equivalent to a couple. Find its moment. [C. H. 1962]

Hint. See Ex. 8. worked.

13. If a variable system of forces in a plane have constant moments about two fixed points in the plane, prove that the resultant passes through another fixed point. [London]

Hint. Reduce w.r. to one of the points as base and line joining them as x-axis.

14. Show that three coplanar forces P, Q, R acting at the points A, B, C are in *astatic* equilibrium if they meet at a point on the circumcircle of the triangle ABC and if $P : Q : R = a : b : c$, where a, b, c are the sides of the triangle ABC. [C. H. 1964]

Hint. By converse of Lami's theorem, show that the forces are in equilibrium. Then follow the argument given in Art. 10.

15. Any number of forces P, Q, R, S,......in one plane are in equilibrium, and their lines of action meet in one point O. Through O, a circle is described, meeting the lines of action of the forces in A, B, C, D,.... If these points are regarded as the points of application of the forces, prove that the equilibrium is astatic.

1·11. Smooth and rough bodies :

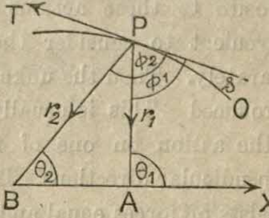
When two bodies are in contact at a point e.g. when a particle rests on a curve or on a surface under the action of external forces, there is always a mutual reaction between the two, and this reaction may or may not act along the common normal at the point of contact. When it acts along the common normal the bodies are said to be *smooth* and when it acts in a direction inclined to the common normal the bodies are said to be *rough*. The component along the common normal is called the *normal reaction* and the tangential component is called the *force of friction*. The reaction is a *passive* force, it comes into play only to oppose the motion and is no more than is necessary to maintain equilibrium.

1·12. Theorem. *A particle rests on a smooth plane curve under the action of given forces. To find the positions of equilibrium.*

It is here assumed that the external forces press the particle against the curve. Since the curve is smooth the only force of reaction it can exert on the particle is along the common normal at the point the particle rests and, therefore, for equilibrium the resultant of the external forces must be also along the common normal but opposite to the force of reaction. Therefore the sum of the resolved parts of the external forces along the tangent to the curve at the point must be zero.

If the external forces be $P_1, P_2, \ldots$ making angles $\theta_1, \theta_2, \ldots$ with the tangent, we must have $P_1\cos\theta_1 + P_2\cos\theta_2 + \cdots = 0$. This equation of condition together with the equation of the curve determines the position or positions of equilibrium.

Ex. A particle P is acted upon by forces $\frac{\mu}{r_1}$ and $\frac{\mu}{r_2}$ towards two fixed points A and B; show that it will rest at any point of a smooth groove whose equation is $r_1 r_2 = c$, where $PA = r_1$ and $PB = r_2$.



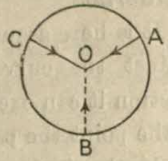
Let PT be the tangent at P and let arc $OP = s$, where O is a fixed point on the curve. Let $\angle PAX = \theta_1$, $\angle PBX = \theta_2$ and let AP, BP make with tangent PT angles ϕ_1 and ϕ_2 respectively. Then resolving the external forces along PT, we must have, for equilibrium, $\frac{\mu}{r_1}\cos\phi_1 + \frac{\mu}{r_2}\cos\phi_2 = 0$

But $\cos\phi_1 = \frac{dr_1}{ds}$ and $\cos\phi_2 = \frac{dr_2}{ds}$ $\therefore$ $\frac{\mu}{r_1}\frac{dr_1}{ds} + \frac{\mu}{r_2}\frac{dr_2}{ds} = 0$

or $r_2\,dr_1 + r_1\,dr_2 = 0$

Differentiating the eq. of the curve, we have, at every point of the curve, $r_2\,dr_1+r_1\,dr_2=0$. Thus the condition is satisfied at every point of the curve and the result follows.

1·13. Action at a joint in a framework : Method of separation of bars.



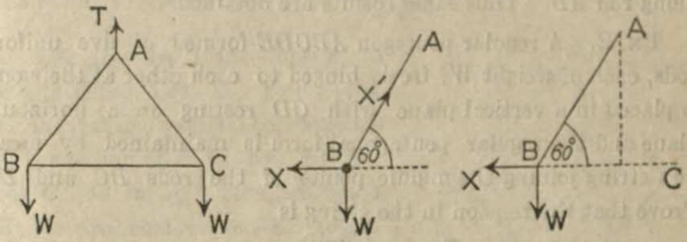
In some problems we have to consider the equilibrium of frameworks formed by uniform bars lying in the same plane jointed together by cylindrical pins passing through small cylindrical holes in the bars and acted upon by given coplanar forces. We shall only consider the case when the pins are *smooth* and the rods are said to be 'smoothly hinged' or 'smoothly jointed'. Take one of the pins connecting two rods P and Q and let the figure show the section of the pin by the plane of the forces. The rod P is in contact with the pin at points A, B, C, ...and the forces of action at these points of contact all act normally to the cylindrical surface of the pin and, therefore, pass through the centre O of the pin and reduce to a *single force* acting at O *i.e.* at the joint which is very small. Thus the action on the rod P at the joint is a single force acting at the joint. Similarly the action on the rod Q at the joint is also a single force acting at the joint and the two actions at the joint are equal and opposite, because the pin, assumed to be weightless, is in equilibrium under the two forces which are equal and opposite to these actions. In solution of problems it is often convenient to consider the equilibrium of one or more bars separately. Then the unknown actions at the joints must be introduced. This is usually done by assuming the resolved parts of the action on one of the rods at each joint in two suitable perpendicular directions. The action at the joint on the other rod consists of forces equal and opposite to these. For convenience, the rods are drawn disjointed in the figure with the actions at the joints shown properly. For equilibrium of the rod or rods considered, the two equations of resolution and the equation of moment are written down. This method is known as the *method of separation of bars.*

The following rules will be helpful in the solution of problems :

(*a*) When there are no external forces acting on a rod then the actions at the two joints at its ends are two equal and opposite forces acting along the rod.

(*b*) When a framework and the external forces which act on it are both symmetrical about any line passing through a joint, then the action at that joint is perpendicular to the line of symmetry.

Ex. 1. Three light equal rods freely jointed to form an equilateral triangle have two equal weights W attached to two joints and the whole is supported at the third joint by means of a string. Find the actions at the joints.



Let us assume that the rods are connected by pins separate from the rods and the wts, W, W are hung from the pins at B and C and the whole triangle ABC is suspended by the pin at A. Then the actions at the joints act along the rods. Let the action of the pin at B on rod BC be X along BC and on rod BA be X' along AB. Then the actions of the rods BC and BA on pin at B are X along CB and X' along BA. Also wt. W is attached to pin at B. These three forces are in equilibrium. By Lami's theorem we have,

$$\frac{X}{\sin(90^\circ+60^\circ)}=\frac{X'}{\sin 90^\circ}=\frac{W}{\sin(180^\circ-60^\circ)} \quad or \quad \frac{X}{\frac{1}{2}}=\frac{X'}{1}=\frac{W}{\frac{\sqrt{3}}{2}}.$$

$$\therefore \quad X=\frac{W}{\sqrt{3}} \text{ and } X'=\frac{2W}{\sqrt{3}}.$$

$\therefore$ Actions along the rods BC, CA, AB are $\frac{W}{\sqrt{3}}, \frac{2W}{\sqrt{3}}, \frac{2W}{\sqrt{3}}.$

Alternatively, we may assume that the pins at B and C are parts of the rods AB and AC and rod BC has holes at B and C into which are fitted the pins. Then the action of rod AB on BC is a force X along BC and therefore action of BC on AB is an equal and opposite force X along CB. The rod AB is in equilibrium under the forces X along CB, wt. W at B and the unknown reaction at A. Taking moments about A, we get,

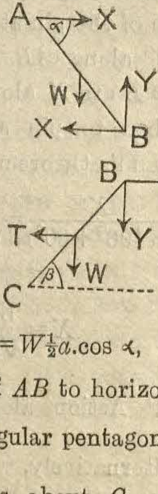
$$X.AB.\sin 60^\circ = W.AB.\cos 60^\circ.$$

or, $X=\frac{W}{\sqrt{3}}$. And the force acting at end B of the rod AB is the resultant of X and W and is, therefore, equal to $\sqrt{W^2+\frac{W^2}{3}}$ $=\frac{2W}{\sqrt{3}}$ acting at an angle $\tan^{-1}\frac{W}{X}=\tan^{-1}\sqrt{3}=60^\circ$ to horizon *i.e.* along rod AB. Thus same results are obtained.

Ex. 2. A regular pentagon $ABCDE$ formed of five uniform rods, each of weight W, freely hinged to each other at their ends is placed in a vertical plane with CD resting on a horizontal plane and the regular pentagonal form is maintained by means of a string joining the middle points of the rods BC and DE. Prove that the tension in the string is

$$\left(\cot\frac{\pi}{5}+3\cot\frac{2\pi}{5}\right).W. \qquad \text{[London]}$$

We shall consider the separate equilibrium of the rods AB and BC only. The action at A of rod AB is a single horizontal force X (from symmetry). The action at B of rod AB is a horizontal force X and a vertical force Y as shown in the figure. Action at B of rod BC are equal and opposite forces X, Y. The wts. W, W of the rods act at their middle points. Also at middle point of rod BC there is the tension of the string T, which is horizontal. Let $AB=a$. For equilibrium of rod AB, resolving vertically and taking moments about B, we get, $Y=W$ and $X.a\sin\alpha=W\frac{1}{2}a.\cos\alpha$,



or, $X=\frac{1}{2}W\cot\alpha$, where α = inclination of AB to horizontal $=\frac{\pi}{5}$, for each angle of a regular pentagon $=\frac{3\pi}{5}$

For equilibrium of rod BC, taking moments about C, where act unknown forces, we get,

$$T.\tfrac{1}{2}a\sin\beta = X.a\sin\beta+Y.a\cos\beta+W.\tfrac{1}{2}a\cos\beta,$$

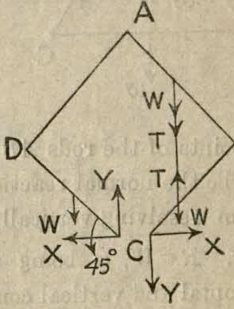
or, $T=2X+(2Y+W)\cot\beta$,

where β = inclination of BC to horizontal $=\frac{2\pi}{5}$

$$\therefore\ T=W\cot\alpha+(2W+W)\cot\beta=W.(\cot\alpha+3\cot\beta)$$

$$=W.\left(\cot\frac{\pi}{5}+3\cot\frac{2\pi}{5}\right).$$

Ex. 3. A square frame $ABCD$ of four equal jointed rods hangs from A, the shape being maintained by a string joining mid-points of AB, BC. Prove that the ratio of the tension of the string to the reaction at C is $\frac{8}{\sqrt{5}}$. [C. H. 1948]



Let $AB=2a$, T the tension in the string. Let X, Y be the horizontal and vertical components of the action at C of rod BC on rod CD as in fig. Then reversed X, Y are components of action at C of rod CD on rod CB.

For equilibrium of CD, taking moments about D, we get,

$$Y.2a\cos 45^\circ=X.2a\sin 45^\circ+W.a\cos 45^\circ$$

$$\text{or,}\quad 2Y=2X+W \quad \dots \quad \dots \quad \dots \quad (1)$$

For equilibrium of CB, taking moments about B,

$$T-W=2X+2Y \quad \dots \quad \dots \quad \dots \quad (2)$$

Lastly consider the equilibrium of the rods AB, BC jointed at B and take moments about A; we get,

$$X.4a\sin 45^\circ=2Wa\cos 45^\circ \text{ or, } X=\tfrac{1}{2}W \quad \dots \quad (3)$$

$\therefore$ From (1), $Y=W$ and from (2), $T=4W$.

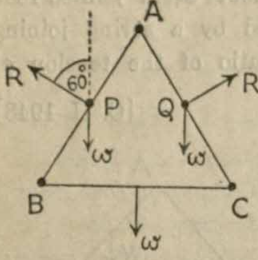
If R be the action at C on CB, then $R=\sqrt{X^2+Y^2}=\frac{W}{2}\sqrt{5}$

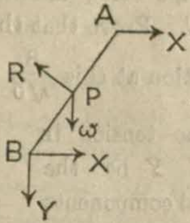
at an angle $\tan^{-1}\frac{Y}{X}=\tan^{-1}2$ with the horizon,

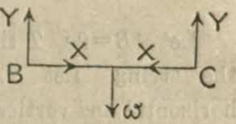
$$\text{and } \frac{T}{R}=\frac{4W}{\frac{1}{2}W\sqrt{5}}=\frac{8}{\sqrt{5}}.$$

Ex. 4. Three equal uniform rods are freely jointed at their extremities and rest in equilibrium over two smooth pegs, in a horizontal line at a distance apart equal to half the length of one rod. If the lowest side be horizontal, then the resultant action at the upper joint is $\frac{5}{18}\sqrt{3}.W$ and at each of the lower $\frac{1}{18}\sqrt{57}W$, where W is the total weight of the rods. [London]

Let AB, BC, CA be the equal rods forming the equilateral triangle ABC and let P, Q be the pegs which evidently are at







mid points of the rods AB and AC, PQ being half of BC. Let R, R be the normal reactions at P, Q. From equilibrium of the system, resolving vertically, we get, $2R\cos 60^\circ = 3\omega = W$

or, $R=W$, ω being wt. of each rod. Let X, Y be the horizontal and vertical components of the action at B of BC on AB, as in fig. Then reversed forces X, Y are components of the action at B of AB on BC. The action at A on rod AB is a horizontal force X', from symmetry. For equilibrium of rod AB, taking moments about A, we get,

$$X.2a\sin 60^\circ + Y.2a\cos 60^\circ + \omega.a\cos 60^\circ = R.a, \quad [AB=2a].$$

$$or,\quad X\sqrt{3}+Y=R-\frac{\omega}{2}=3\omega-\frac{\omega}{2}=\tfrac{5}{2}\omega. \quad \ldots \quad (1)$$

And resolving horizontally, we get, $R\sin 60^\circ = X+X'$ ... (2)

Finally, for equilibrium of BC, the action at C must have horizontal and vertical components X, Y as in fig., since ω acts at mid. point of BC (or from symmetry) ; and resolving vertically $2Y=\omega$.

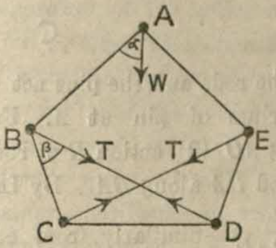
$$\therefore\quad Y=\frac{\omega}{2} \text{ and from (1), } X=\frac{2}{\sqrt{3}}\omega \text{ and from (2), } X'=\frac{5}{2\sqrt{3}}\omega.$$

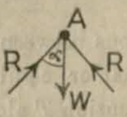
$$\therefore\quad R=W,\; X=\frac{2}{3\sqrt{3}}W \text{ and } Y=\frac{1}{6}W \text{ and } X'=\frac{5}{6\sqrt{3}}W.$$

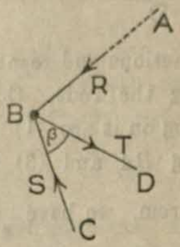
Hence action at A is a horizontal force $\frac{5}{18}\sqrt{3}W$ and action at B or C is $\sqrt{X^2+Y^2}=\frac{1}{18}\sqrt{57}W$ at an angle $\tan^{-1}\frac{\sqrt{3}}{4}$ with horizontal.

Ex. 5. Five weightless rods form a regular pentagon $ABCDE$ and the framework is stiffened by other weightless rods BD, CE. The system is placed in a vertical plane with CD on a horizontal table, and a weight W is hung from A. Prove that the thrust in BD or CE is $W\cot\frac{\pi}{5}$. [C. H. 1951]

We assume that the rods are joined by distinct pins. Then their actions on the rods are along the respective rods. Let the actions of the rods AB, AE on the pin at A be R, R along BA and EA. Then for equilibrium of pin A, resolving vertically, we get $2R\cos\alpha=W$, where $2\alpha=$ angle in a regular pentagon $=\frac{3\pi}{5}$.







Consider equilibrium of pin at B. Forces acting on it are (1) action R of rod AB acting along AB (2) action S of rod CB acting along CB and (3) tension T of rod BD acting along BD. By Lami's theorem, $\frac{R}{\sin\beta}=\frac{T}{\sin\frac{3\pi}{5}}$, where $2\beta=\pi-\frac{3\pi}{5}$ *or*, $\beta=\frac{\pi}{5}$.

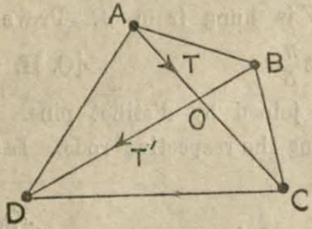
$$\therefore\quad T=\frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}}\cdot\frac{W}{2\cos\frac{3\pi}{10}}=\frac{W}{\sin\frac{\pi}{5}}\cdot\sin\frac{3\pi}{10}$$

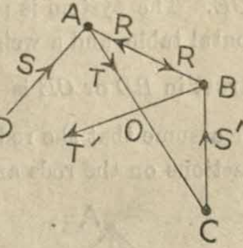
$$=\frac{W}{\sin\frac{\pi}{5}}\cdot\cos\frac{\pi}{5}=W\cot\frac{\pi}{5}$$

Ex. 6. Four rods are smoothly jointed at their extremities to form a quadrilateral $ABCD$, and the opposite corners A, C

and B, D are joined by tight strings. If T, T' be the tensions in AC and BD, show that $T\left(\frac{1}{AO}+\frac{1}{OC}\right)=T'\left(\frac{1}{BO}+\frac{1}{OD}\right)$. where O is the intersection of the diagonals. [P. C. S, 1940]

We shall assume the rods to be pin-jointed by pins distinct from the rods and the strings to be attached to the pins. Then





the actions and reactions between the rods and the pins act only along the rods. Consider equilibrium of pin at A. Forces acting on it are (1) tension T along AO (2) action R of rod BA along BA and (3) action S of rod DA along DA. By Lami's theorem, we have, $\frac{T}{\sin BAD}=\frac{R}{\sin OAD}$. Similarly from equilibrium of pin at B, we have,

$$\frac{T'}{\sin ABC}=\frac{R}{\sin OBC} \quad \therefore \quad T.\frac{\sin OAD}{\sin BAD}=T'.\frac{\sin OBC}{\sin ABC}$$

Now from $\triangle BAD$ and $\triangle ABC$, we have,

$$\frac{\sin BAD}{\sin BDA}=\frac{BD}{AB} \text{ and } \frac{\sin ABC}{\sin BCA}=\frac{AC}{AB}.$$

Hence $T.\frac{\sin OAD}{\sin ODA}.\frac{AB}{BD}=T'.\frac{\sin OBC}{\sin BCO}.\frac{AB}{AC}$.

$$or, \quad T.\frac{OD}{OA}.\frac{1}{BD}=T'.\frac{OC}{OB}.\frac{1}{AC}.$$

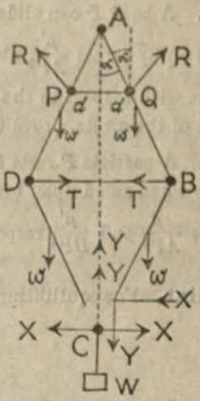
$$or, \quad T.\frac{AC}{OA.OC}=T'.\frac{BD}{OB.OD}$$

$$or, \quad T\left(\frac{1}{AO}+\frac{1}{OC}\right)=T'\left(\frac{1}{BO}+\frac{1}{OD}\right).$$

Ex. 7. A rhombus $ABCD$, formed of four uniform rods, each of length l and weight w, joined by smooth hinges, rests in equilibrium with AC vertical on two smooth pegs in a horizontal line distance $2d$ apart, a weight W is attached to the lowest hinge C, and B, D are joined by a string. If the angle at A is 2α, show that the tension of the string is

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+4w)\right].\qquad\text{[C. H. 1966]}$$

Let P, Q be the pegs and let R, R be the normal reactions of the pegs on the rods.



For equilibrium of the whole system, resolving vertically, we get, $2R\sin\alpha=W+4w\ldots(1)$. Consider equilibrium of the pin at C. Let X, Y be the horizontal and vertical components of the action of rod BC on pin C, then from symmetry, X, Y are also the horizontal and vertical components of the action of rod DC on pin C, as shown in fig. Resolving vertically we get $2Y=W\ldots(2)$. Next consider equilibrium of rod CB. At C, the reaction of pin C on CB has horizontal and vertical components X, Y reversed, as in fig. Weight of CB is w acting at its middle point and unknown reaction acts at B. Taking moment about B, we get,

$$Xl\cos\alpha=Yl\sin\alpha+w.\frac{l}{2}\sin\alpha.$$

$$\text{or,}\quad 2X\cos\alpha=(W+w)\sin\alpha\ldots\quad\ldots(3)\quad[\text{ from }(2)]$$

Finally, for equilibrium of rods AB, BC jointed at B, taking moments about A, we get,

$$Rd\operatorname{cosec}\alpha=X.2l\cos\alpha+2w.\frac{l}{2}\sin\alpha+T.l\cos\alpha.$$

$$\text{or,}\quad 2Tl\cos\alpha=(W+4w)d\operatorname{cosec}^2\alpha-2l(W+w)\sin\alpha-2lw\sin\alpha$$
$$[\text{ from }(1),(3)].$$

$$\therefore\quad T=\frac{\sin\alpha}{2l\cos\alpha}\left[(W+4w)d\operatorname{cosec}^3\alpha-2l(W+2w)\right]$$

$$=\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

EXAMPLES

1. A bead rests on a smooth wire in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and is acted on by forces kx^3, ky^3 parallel to the axes. Find the positions of equilibrium.

Hint. For equilibrium, $kx^3\frac{dx}{ds}+ky^3\frac{dy}{ds}=0$ *or*, $x^3+y^3\frac{dy}{dx}=0$ etc.

2. A bead P can slide on a smooth circular wire of centre O, and is acted on by forces $\frac{\mu'}{AP^2}$ along PA and $\frac{\mu}{BP^2}$ along BP where A, B are inverse points with respect to the circle. If the bead be in equilibrium at every point of the circle, prove that $\mu'^2 : \mu^2 = OA : OB$.

3. A particle P rests in equilibrium at every point of a smooth plane curve when acted upon towards two centres in the plane at A and B by forces $\frac{\mu}{AP^2}$ and $\frac{\mu'}{BP^2}$ respectively. Find the equation of the curve.

Hint. For equilibrium, $\frac{\mu}{r^2}\frac{dr}{ds}+\frac{\mu'}{r'^2}\frac{dr'}{ds}=0$, where $AP=r$, $BP=r'$.

$$\therefore\ \frac{u}{r}+\frac{\mu'}{r'}=k, \text{ is the eqn.}$$

4. A small bead P can slide on a smooth elliptic wire; it is attracted towards the foci S and H by forces proportional to $(SP)^m$ and $(HP)^n$ respectively; find the positions of equilibrium.

5. Three equal uniform rods, each of weight W, are smoothly jointed so as to form an equilateral triangle. If the system is supported at the middle point of one of the rods, show that the action at the lowest angular point is $\frac{W}{6}\sqrt{3}$, and that at each of the others is $W\sqrt{\frac{13}{12}}$ acting at an angle $\tan^{-1}(2\sqrt{3})$ to the horizon. [C. U. 1965]

6. Three equal uniform rods, each of weight W, are smoothly pin-jointed to form an equilateral triangle ABC which hangs in equilibrium from A. Find the actions of the rods on the pins.

Hint. Consider separate equilibrium of BC and AC.

7. Two rods AB, BC of equal weight but of unequal length, are hinged together at B, and their extremities are attached to two fixed hinges A and C in the same vertical line. Prove that the line of action of the reaction at the hinge B bisects the straight line AC.

8. Three rods, jointed together at their extremities, are laid on a smooth horizontal table; and forces are applied at the middle points of the sides of the triangle formed by the rods, and respectively perpendicular to them. Show that, if these forces produce equilibrium, the actions at the joints will be all equal and their directions will touch the circumcircle of the triangle. [M.T.]

9. Four rods, freely jointed at their extremities A, B, C, D form a parallelogram. The opposite corners are joined by strings along the two diagonals, each of which is tight. Show that their tensions are proportional to the diagonals along which they act.

10. Four equal heavy uniform rods AB, BC, CD, DA are jointed to form a rhombus and the corners A and C are joined by a string. If the rhombus is suspended by corner A, find the tension of the string and the reaction at B or D.

11. A regular pertagon $ABCDE$, formed of five equal heavy rods jointed together, is suspended from joint A and the regular pentagonal form is maintained by a weightless rod joining the middle points K, L of BC and DE. Prove that stress at K or L is $2W \cot 18°$, where W is the weight of a rod. [M. T.]

12. A regular hexagon $ABCDEF$, made of six equal uniform rods jointed freely, rests in a vertical plane having AB fixed in a given horizontal plane and C, F are connected by a light inextensible string. Find the tension of the string and the actions at the joints E and F. [London]

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FRICTION

2·1. As we have stated in Art. 1·11, when two bodies are in contact at one or more points, then at each point of contact there is a mutual reaction between the two bodies. The component of this mutual reaction lying in the tangent plane at the point of contact is called *Friction*. Friction is a *passive force*. It comes into existence only when required to prevent the relative motion of the points of contact *e.g.* sliding of one body over another. It comes into play only to *prevent* and *not to cause* the motion. From various experiments, the following laws, known as laws of friction, have been deduced.

Laws of Friction :

(i) Friction acts in the direction opposite to the direction in which the point of contact is urged to move.

(ii) The magnitude of the friction is just sufficient to prevent the motion of the point of contact.

(iii) The amount of friction that can be called into play cannot exceed a certain maximum amount, called *limiting friction*.

(iv) The magnitude of limiting friction bears a constant ratio to the normal reaction between the bodies. If R be the normal reaction then limiting friction is μR, where μ is a constant, depending only on the materials of which the bodies are made.

(v) Limiting friction is independent of the areas in contact provided the normal reaction is unaltered.

(vi) When motion takes place, the above laws of limiting friction are still true but the coefficient of friction is slightly less when the bodies are in relative motion than when the bodies are at rest. Also the friction is independent of velocity.

Observations on the laws :

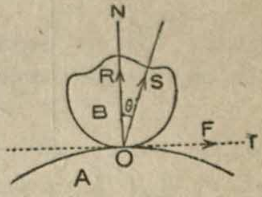
Law (i). To decide whether there is friction at the point of contact of two bodies, imagine the bodies to be smooth at the point of contact ; if on this supposition the point of contact would move then there is friction at the point of contact and its direction is opposite to that in which motion would ensue.

Law (ii). This law shows that friction is a self-adjusting force and no more of it is called into play than is sufficient to prevent motion.

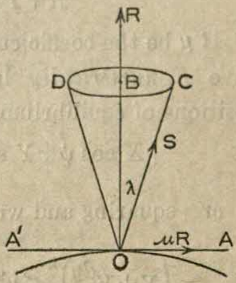
Law (iii). This law states that the amount of friction that can come into existence is not, however, unlimited. If for maintaining equilibrium, a greater amount of friction than its maximum value be necessary, then equilibrium will not be maintained and motion will take place If follows that for equilibrium, friction $F \ngtr \mu R$. When $F=\mu R$ the equilibrium is called *limiting equilibrium.*

2·2. Angle of Friction, Cone of friction.

When two bodies in contact are in limiting equilibrium the angle which the resultant reaction at the point of contact makes with the normal at the point is called the *angle of friction.*



Let O be the point of contact and S the resultant reaction and λ the angle it makes with the normal. Then normal reaction $R=S\cos\lambda$, and friction $F=S\sin\lambda$. But as friction is limiting, $F=\mu R$ $\therefore$ $\tan\lambda=\mu$. or *coefficient of friction is equal to the tangent of the angle of friction.* When equilibrium is not limiting, let S make with the normal an angle θ, as in fig. Then $R=S\cos\theta$, $F=S\sin\theta$. But $F \ngtr \mu R$ $\therefore$ $\tan\theta \ngtr \mu=\tan\lambda$ $\therefore$ $\theta \ngtr \lambda$. Thus the *resultant reaction can never make with the normal an angle greater than the angle of friction, but can make any smaller angle, if necessary for preserving equilibrium.*

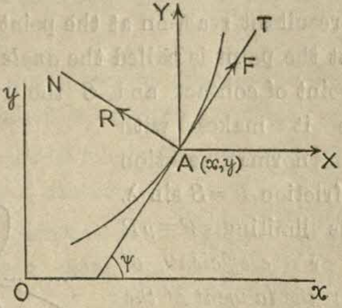


Suppose two bodies are in contact at O; draw the common normal OB at O and with O as vertex and OB as axis construct the right circular cone of semi-vertical angle λ, the angle of friction. Then the resultant reaction at O can have any direction lying within or upon this cone but it cannot have a direction which lies outside the cone. This cone is called the *cone of friction.*

This gives a method of finding the conditions of equilibrium of a body resting on another body at O under the action of external forces. Since for equilibrium the resultant reaction at O is equal and opposite to the resultant of the external forces it follows that for equilibrium the external forces must be reducible to a *single force at O whose line of action falls within the cone of friction.*

2·3. Theorem. *A particle is constrained to rest on a rough plane curve under the action of any given forces. To find the positions of equilibrium.*

Suppose the particle is at rest at $A\ (x, y)$ and X, Y are the components of the given forces parallel to the co-ordinate axes Ox, Oy. The particle is supposed to be pressed against the curve by the given forces. Let R be the normal reaction of the curve on the particle along AN, the inward normal, and F the friction called into play along the tangent AT at A which makes an angle ψ with Ox. Resolving along AT and AN, we have,



$$X \cos\psi + Y \sin\psi + F = 0$$

$$R + Y \cos\psi - X \sin\psi = 0$$

If μ be the coefficient of friction, for equilibrium we must have F *numerically* less than or equal to μR. Therefore the positions of equilibrium are those positions where

$$|X \cos\psi + Y \sin\psi| \leqslant \mu\, |X \sin\psi - Y \cos\psi|$$

or squaring and writing $\dfrac{dy}{dx}$ for $\tan\psi$,

$$\left(X + Y\frac{dy}{dx}\right)^2 \leqslant \mu^2 \left(X\frac{dy}{dx} - Y\right)^2.$$

The sign of equality gives the positions of limiting equilibrium.

2·4. If in Art 2·3, the curve be a space curve and (x, y, z) be the point A and X, Y, Z the components of the given forces parallel to axes Ox, Oy, Oz, the direction cosines of the tangent AT are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ and those of the resultant force are proportional to X, Y, Z. If θ be the angle the resultant force makes with the normal at A, then $90-\theta$ is the angle it makes with AT.

$$\therefore \quad \cos(90-\theta)=\frac{\Sigma X\frac{dx}{ds}}{\sqrt{X^2+Y^2+Z^2}}.$$

But for equilibrium $\theta \leqslant \lambda$. $\therefore$ $\tan\theta \leqslant \tan\lambda=\mu$

$$\text{or} \quad \sin^2\theta \leqslant \mu^2(1-\sin^2\theta). \quad \text{or} \quad \sin^2\theta \leqslant \frac{\mu^2}{1+\mu^2}$$

$\therefore$ For positions of equilibrium, we must have,

$$\left(\Sigma X\frac{dx}{ds}\right)^2 \leqslant \frac{\mu^2}{1+\mu^2}(X^2+Y^2+Z^2).$$

2·5. **Theorem.** *A particle is constrained to rest on a given rough surface under the action of any given forces. To find the positions of equilibrium.*

Let the equation of the given surface be $f(x, y, z)=0$ and let the particle be at rest at $A(x, y, z)$ and let X, Y, Z be the components of the given forces parallel to the co-ordinate axes. The direction cosines of the normal at A are proportional to f_x, f_y, f_z and those of the resultant of the given forces are proportional to X, Y, Z. If θ be the angle the resultant force makes with the normal at A, we have

$$\cos\theta=\frac{Xf_x+Yf_y+Zf_z}{\sqrt{X^2+Y^2+Z^2}\sqrt{f_x^2+f_y^2+f_z^2}}.$$

For equilibrium, we have $\theta \leqslant \lambda$ or $\tan\theta \leqslant \tan\lambda=\mu$

$$\therefore \quad \sec^2\theta=1+\tan^2\theta \leqslant 1+\mu^2$$

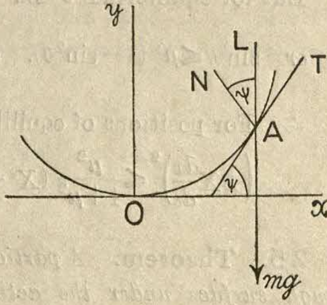
$$\text{or} \quad \cos^2\theta \geqslant \frac{1}{1+\mu^2}.$$

Hence for positions of equilibrium, we must have,

$$(\Sigma Xf_x)^2 \geqslant \frac{1}{1+\mu^2}. \quad \Sigma X^2. \quad \Sigma f_x^2. \cdots\cdots(1)$$

Here X, Y, Z, f are all functions of x, y, z. The equality sign in (1) gives the equation of a surface whose curve of intersection with given surface $f(x, y, z)=0$ is the boundary of the positions of eqilibrium of the particle on given surface.

Ex. 1. A cycloid is placed with its axis vertical and vertex downwards. Show that a particle cannot rest at any point of the curve which is higher than $2a \sin^2\lambda$ above its lowest point where λ is the angle of friction and a is the radius of the generating circle of the cycloid. [C. H. 1967]



Let O be the lowest point of the cycloid, and let the tangent and the normal at O be the axes Ox, Oy. The parametric equation of the cycloid is

$$\left.\begin{aligned} x&=a(\theta+\sin\theta) \\ y&=a(1-\cos\theta) \end{aligned}\right\} \quad \cdots \quad \cdots \quad (1)$$

Let the particle be at rest at A whose parametric coordinate is 'θ' and let AT, the tangent at A make an angle ψ with Ox. Then $\psi=\frac{1}{2}\theta$. The only external force acting on the particle is its weight mg acting vertically down along LA. Therefore the resultant reaction at A which balances it must act along AL. For equilibrium AL must make with normal AN an angle less than or equal to λ. But $\angle LAN=\psi$

$$\therefore \ \psi \leqslant \lambda \quad \ldots \quad \ldots \quad (2)$$

Now height of A above Ox is $y=a(1-\cos\theta)=2a\sin^2\dfrac{\theta}{2}$

$$=2a\sin^2\psi$$

$\therefore$ for equilibrium to be possible, we must have,

$$y=2a\sin^2\psi \leqslant 2a\sin^2\lambda. \quad \text{(from 2)}$$

So the particle cannot rest at a point whose height above O is greater than $2a\sin^2\lambda$.

Ex. 2. A rough wire which has the shape of an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, is placed with it x-axis vertical and y-axis horizontal. If μ be the coefficient of friction, find the depth below the highest point of the position of limiting equilibrium of a bead which rests on the wire. [C. H. 1956, 1954]

For positions of limiting equilibrium, we have

$$\left(X+Y\frac{dy}{dx}\right)^2=\mu^2\left(X\frac{dy}{dx}-Y\right)^2 \qquad \text{(Art. 2·3)}$$

Here $Y=0,\ X=mg,\ \therefore\ 1=\mu^2.\left(\frac{dy}{dx}\right)^2$.

Differentiating equation of curve, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0 \quad or, \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

$$\therefore\quad 1=\mu^2\cdot\frac{b^4x^2}{a^4y^2} \quad or, \quad \mu^2b^4x^2=a^4.b^2\left(1-\frac{x^2}{a^2}\right) \quad \text{[from eqn. of curve]}$$

$$or,\quad x^2(a^2+\mu^2b^2)=a^4$$

$$or,\quad x=\frac{a^2}{\sqrt{a^2+\mu^2b^2}}$$

$\therefore$ x co-ordinate of limiting position of equilibrium is $\frac{a^2}{\sqrt{a^2+\mu^2b^2}}$. Its depth below highest point *i.e.*, upper end of major-axis is $\left(a-\frac{a^2}{\sqrt{a^2+\mu^2b^2}}\right)$.

Alt. Let $P(a\cos\theta,\ b\sin\theta)$ be the position of limiting equilibrium. Equation of normal at P is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$$

$\therefore$ slope of normal$=\frac{a}{b}\tan\theta$. But in position of limiting equilibrium, the vertical weight which is parallel to the x-axis and is the only external force acting on the particle, makes with the normal the angle λ.

$\therefore$ $\frac{a}{b}\tan\theta=\tan\lambda=\mu$. This determines the eccentric angle of P, the position of limiting equilibrium.

Now, height of P above centre of ellipse

$$=a\cos\theta=\frac{a}{\sqrt{1+\tan^2\theta}}=\frac{a}{\sqrt{1+\frac{b^2\mu^2}{a^2}}}=\frac{a^2}{\sqrt{a^2+b^2\mu^2}}$$

$$\therefore\ \text{Reqd. depth}=a-\frac{a^2}{\sqrt{a^2+b^2\mu^2}}.$$

Ex. 3. The paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ is placed with its axis vertical and its vertex uppermost; if μ be the coefficient of friction, show that a particle will rest on it at any point above its curve of intersection with the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\mu^2$.

Here $f(x, y, z)\equiv\frac{x^2}{a}+\frac{y^2}{b}-2z=0$

$\therefore\ f_x=\frac{2x}{a},\ f_y=\frac{2y}{b},\ f_z=-2$; Also $X=0,\ Y=0,\ Z=mg$

For equilibrium, $(\Sigma Xf_x)^2\geqslant\frac{1}{1+\mu^2}\Sigma X^2.\Sigma f_x^2$

$$\therefore\ [mg.(-2)]^2\geqslant\frac{1}{1+\mu^2}\cdot(mg)^2\cdot\left(\frac{4x^2}{a^2}+\frac{4y^2}{b^2}+\ \right)$$

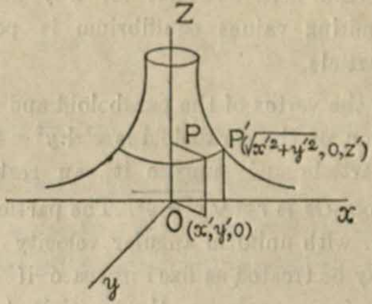
$$or,\ 1+\mu^2\geqslant\frac{x^2}{a^2}+\frac{y^2}{b^2}+1\quad or,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant\mu^2.$$

$\therefore$ The particle will rest in equilibrium at any point on the surface above its curve of intersection with the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\mu^2$, which is an elliptic cylinder with generators parallel to the axis of the paraboloid.

Ex. 4. A rough surface is formed by the revolution of a rectangular hyperbola about a vertical asymptote; show that a particle will rest on it anywhere beyond its intersection with a certain circular cylinder. [C. H. 1950]

The vertical asymptote is taken as the z-axis and the other as x-axis. Equation of generating hyperbola is $xz=c^2$ ... (1)

Let $P(x', y', z')$ be any point of the surface formed by revolution of curve (1) about Oz. Distance of P from $Oz = \sqrt{x'^2+y'^2}$.



P is on the circle generated by revolution of point P' of curve (1) whose co-ordinates are, therefore, $x = \sqrt{x'^2+y'^2}$, $z = z'$. Since P' is on curve (1)

$\therefore\ \sqrt{x'^2+y'^2}.z' = c^2$

$\therefore$ Eqn. of surface generated is $(x^2+y^2)z^2 = c^4$. ... (2)

Here $f(x, y, z) \equiv (x^2+y^2)z^2 - c^4 = 0$

$\therefore\ f_x = 2xz^2,\ f_y = 2yz^2,\ f_z = (x^2+y^2).2z$. And $X=0$, $Y=0$, $Z=-mg$.

For equilibrium, $(\Sigma X f_x)^2 \geqslant \dfrac{1}{1+\mu^2}\Sigma X^2.\Sigma f_x^2$.

$\therefore\ (1+\mu^2).f_z^2 \geqslant \Sigma f_x^2$ *or,* $\mu^2.4z^2(x^2+y^2)^2 \geqslant 4x^2z^4 + 4y^2z^4$

or, $\mu^2(x^2+y^2) \geqslant z^2$. ... (3)

Limiting positions of equilibrium are given by the curve of intersection of surface (2) and the right circular cone $\mu^2(x^2+y^2) = z^2$ whose vertex is O and axis Oz and the particle will rest anywhere on the surface (2) which is outside this cone. The curve of intersection also satisfies the eqn. $\mu^2(x^2+y^2)^2 = c^4$ (eliminating z).

or, $x^2+y^2 = \dfrac{c^2}{\mu}$ which is a right circular cylinder. Hence result follows.

Ex. 5. A rough paraboloid of revolution, of latus rectum $4a$ and of co-efficient of friction $\cot\beta$, revolves with uniform angular velocity ω about its axis which is vertical ; if $\omega > \sqrt{\dfrac{g}{2a}}\cot\dfrac{\beta}{2}$

or, $< \sqrt{\frac{g}{2a}}\tan\frac{\beta}{2}$, show that a particle can rest anywhere except within a certain belt, but that for any angular velocity between these limiting values equilibrium is possible for all positions of the particle. [M. T.]

Take origin at the vertex of the paraboloid and its axis as the axis of z. Equation of the paraboloid is $x^2+y^2=4az$. Let m be the mass of the particle and suppose it can rest at $P(x, y, z)$ whose distance from Oz is $r=\sqrt{x^2+y^2}$. The particle is describing a circle of radius r with uniform angular velocity ω. Therefore the paraboloid may be treated as fixed in space if we apply the reversed effective force $m\omega^2 r$ on the particle tending directly away from the axis. Then, we have $f(x, y, z)\equiv x^2+y^2-4az=0$, $X=m\omega^2x$, $Y=m\omega^2y$, $Z=-mg$.

$\therefore \quad f_x=2x, f_y=2y, f_z=-4a$

$\therefore$ For equilibrium, $(\Sigma X f_x)^2 \geqslant \frac{1}{1+\mu^2}\cdot\Sigma X^2 . \Sigma f_x^2$

or, $(\omega^2x^2+\omega^2y^2+2ag)^2(1+\mu^2)\geqslant(x^2+y^2+4a^2)(\omega^4x^2+\omega^4y^2+g^2)$

or, writing r^2 for x^2+y^2 and simplifying, we get,

$$\mu^2(\omega^2r^2+2ag)^2-r^2(2a\omega^2-g)^2\geqslant 0,$$

or, $[\mu\omega^2.r^2-(2a\omega^2-g)r+2a\mu g][\mu\omega^2r^2+(2a\omega^2-g)r+2a\mu g]\geqslant 0$ ... (1)

Consider the eqns. $\mu\omega^2r^2-(2a\omega^2-g)r+2a\mu g=0$ ... ... (2)

and $\mu\omega^2r^2+(2a\omega^2-g)r+2a\mu g=0$ ... ... (3)

If r_1, r_2 are roots of (2), $-r_1$, $-r_2$ are roots of (3). Also r_1, r_2 have same sign. If roots of one eqn. be imaginary, the roots of the other are also imaginary, and then condition (1) holds for all values of r. If r_1, r_2 are real and distinct, then condition (1) can be written,

$$\mu^2\omega^4(r-r_1)(r-r_2)(r+r_1)(r+r_2)\geqslant 0.$$

We may assume $r_2>r_1>0$. Then the condition holds if $r<r_1$ *or,* $r>r_2$ and does not hold if $r_1<r<r_2$, Thus if r_1, r_2 are imaginary, particle rests anywhere on the surface but if r_1, r_2 are real then there is the belt $r_1<r<r_2$, where particle cannot rest but it can rest anywhere outside the belt.

Now r_1, r_2 will be real or imaginary according as,

$(2a\omega^2-g)^2-8a\mu^2\omega^2g\gtreqless 0$ *or,* $4a^2\omega^4-4ag(1+2\mu^2)\omega^2+g^2\gtreqless 0$

Roots of $4a^2\omega^4-4ag(1+2\mu^2).\omega^2+g^2=0$ are given by

$$\omega^2=\frac{g}{2a}(\sqrt{1+\mu^2}\pm\mu)^2 \quad or, \quad \omega_1^2,\ \omega_2^2 \text{ where}$$

$$\omega_1=\sqrt{\frac{g}{2a}}(\sqrt{1+\mu^2}-\mu)>0$$

$$\text{and } \omega_2=\sqrt{\frac{g}{2a}}(\sqrt{1+\mu^2}+\mu)>0 \text{ and } \omega_1<\omega_2$$

$\therefore$ r_1, r_2 are real or imaginary acc. as

$$4a^2(\omega-\omega_1)(\omega+\omega_1)(\omega-\omega_2)(\omega+\omega_2)\gtrless 0$$

$\therefore$ r_1, r_2 are real if $\omega<\omega_1$ or $>\omega_2$ and r_1, r_2 are imaginary if $\omega_1<\omega<\omega_2$.

Putting $\mu=\cot\beta$, we find,

$$\omega_1=\sqrt{\frac{g}{2a}}\tan\frac{\beta}{2},\quad \omega_2=\sqrt{\frac{g}{2a}}\cot\frac{\beta}{2}.$$

Hence if $\omega<\sqrt{\frac{g}{2a}}\cdot\tan\frac{\beta}{2}$ or $>\sqrt{\frac{g}{2a}}\cdot\cot\frac{\beta}{2}$, there is a belt where particle cannot rest but can rest anywhere outside the belt and if ω lies between the above two values, the particle can rest everywhere on the surface.

EXAMPLES

1. A heavy particle rests on a rough parabola with its axis vertical and vertex downwards. If the latus rectum of the parabola be $4a$, find the greatest altitude above the vertex at which the particle could remain at rest, the coefficient of friction being μ.

2. A rough wire is in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A small bead on the wire is in limiting equilibrium when at a distance $\frac{1}{2}b$ from the major axis under the action of a force whose line of action passes through the centre of the ellipse. Find the coefficient of friction.

3. A heavy bead of weight W can slide on a rough circular wire fixed in space with its plane vertical. A centre of repulsive force is situated at one extremity of the horizontal diameter, and the force on the bead when at a distance r is pr. Find the limiting positions of equilibrium.

4. A rough wire in the form of a parabola of latus rectum $4a$, with its axis vertical and vertex upwards, rotates with uniform angular velocity $\sqrt{\frac{g}{2a}}$ about its axis. A small ring of mass m can slide on the wire and is attached to the focus by an elastic string of natural length a and modulus

of elasticity $\frac{2}{3}$ mg. Prove that the ring will remain in equilibrium at any point of the wire whose distance from the axis is less than $a\mu$, where μ is the coefficient of friction between the ring and the wire.

Hint. Apply to m the reverse effective force and use Art. 2·3.

5. The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is placed with the axis of x vertical and its surface is rough. Show that a heavy particle will rest on it anywhere above its intersection with the cylinder

$$\frac{y^2}{b^2}\left(1+\frac{a^2}{\mu^2b^2}\right)+\frac{z^2}{c^2}\left(1+\frac{a^2}{\mu^2c^2}\right)=1,$$

where μ is the coefficient of friction. (London)

6. A particle rests on the surface $xyz=c^3$ under the action of a constant force parallel to the axis of z; show that the curve of intersection of the surface with the cone $\frac{1}{x^2}+\frac{1}{y^2}=\frac{\mu^2}{z^2}$ will separate the part of the surface on which equilibrium is possible from that on which it is not possible. (M. T.)

2·6. Problems on friction :

In solving problems involving rough bodies subject to frictional forces two fundamental principles should be remembered : (1) if frictional forces *can* produce equilibrium, they *will* ; (2) frictional forces called into play are always *just sufficient* to prevent motion and *no more.*

In first principle it is implied that the laws of friction *must not be violated.* The direction of the force of friction at a point of contact must be opposite to that in which the point of contact would move if there be no friction and its magnitude *must not be greater than* μR, where R is the normal reaction at the point, but may be less than μR, if that will produce equilibrium. From this follows the important rule—*The resultant reaction at a point of contact can assume whatever magnitude and direction are necessary to produce equilibrium subject only to the condition that its direction must not make with the normal at the point an angle greater than the angle of friction.* The second principle leads to the well-known rule—*if a body can roll it will roll* (*or turn over*) *in preference to sliding.* For in rolling or turning over less friction is called into play than required to prevent sliding.

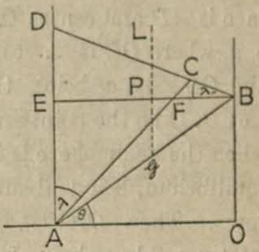
There are various types of problems on friction. Generally all such problems can be solved by the use of above two rules.

We shall illustrate this by solving some such problems. Solution of problems on initial motion involving unknown directions of frictional forces will be taken up later.

Examples Worked

Ex. 1. A ladder is placed with one end on a rough horizontal floor and the other against a rough vertical wall, the ladder being in a plane perpendicular to the wall and inclined at an angle θ to the horizon. Discuss its equilibrium.

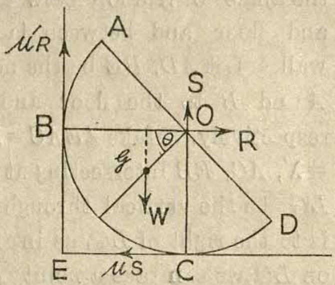
Let AB be the ladder of weight ω and G its c.g. Let λ, λ' be the angles of friction between ladder and floor and between ladder and wall. Let AD, BE be the normals at A and B to the floor and the wall respectively. Make $\angle EAC=\lambda$, $\angle EBD=\lambda'$, AC, EB intersecting at F. Let LG be the vertical through G. If C is to the right of LG, as in fig., then on LG we can take a point P which lies within the quadrilateral $EFCD$ and the resultant reactions at A and B can act along AP and BP and keep the ladder in equilibrium. But if C is to the left of LG, then since the resultant reactions at A and B must intersect on LG to balance ω in order to produce equilibrium, one of them at least must make with the corresponding normal an angle greater than the angle of friction and this being impossible, equilibrium cannot hold and the ladder will slide down. Thus whether there will be equilibrium or not depends on whether the vertical line through the c.g. passes through the quadrilateral area $EFCD$ or it does not.



When C is to the right of LG, a man of weight W can ascend the ladder so long as the line of action of the resultant weight $(\omega+W)$ passes to the left of C and cross the area $EFCD$ but he cannot ascend so high that the said line of action passes to the right of C. Similarly when C is to the left of LG, to keep the ladder in equilibrium, a weight W of sufficient magnitude should be placed on the ladder to the left of G so that the line of action of resultant wt. $(\omega+W)$ may shift to the left of C and cross the area $EFCD$. It should be noted that when equilibrium holds, the reactions at A and B are indeterminate, unless the equilibrium is *limiting* when LG passes *through* C and the resultant reactions acting along AC and BC are determinate in magnitude and direction.

Ex. 2. A solid homogeneous hemisphere rests on a rough horizontal plane and against a rough vertical wall, the coefficients of friction being μ and μ' respectively. Show that the least angle that the base of the hemisphere can make with the vertical is $\cos^{-1}\left(\frac{8\mu}{3}\cdot\frac{1+\mu'}{1+\mu\mu'}\right)$.

Let W be the weight of the hemisphere $ABCD$ whose plane face is AD and centre O. G is its c. g. where OG is $\perp$ to face AD and $OG=\frac{3}{8}a$, a being the radius. Let B, C be the points of contact when the hemisphere is in limiting equilibrium, being about to slide down. Then $\angle BOG=\theta$, has the least possible value. Let R and S be the normal reactions at B and C. $\therefore$ Friction at B is $\mu' R$ acting along EB and that at C is μS along CE. Resolving horizontally and vertically and taking moments about C, we get, $R=\mu S$...(1), $\mu' R+S=W$...(2) $R.a+\mu' R.a=\frac{3}{8}a\cos\theta.W$ *or*, $R(1+\mu')=\frac{3}{8}\cos\theta.W$...(3)



From (1) and (2), $(\mu'\mu+1)R=W\mu$.

$$\therefore\quad (\mu\mu'+1)\tfrac{3}{8}\cos\theta=\mu(1+\mu')\quad[\text{ from (3) }]$$

$$or,\quad \cos\theta=\tfrac{8}{3}\cdot\frac{\mu(1+\mu')}{1+\mu\mu'}.$$

$\therefore$ Least angle plane face can make with vertical

$$=\theta=\cos^{-1}\left(\frac{8\mu}{3}\cdot\frac{1+\mu'}{1+\mu\mu'}\right).$$

Ex. 3. An elliptic cylinder rests in limiting equilibrium between a rough vertical and an equally rough horizontal plane, the axis of the cylinder being horizontal and the major axis of the ellipse inclined to the horizon at an angle of 45° ; shew that the coefficient of friction is $(\sqrt{1+2e^2-e^4}-1)/(2-e^2)$, where e is the eccentricity of the cross-section of the cylinder. [C. H. 1966]

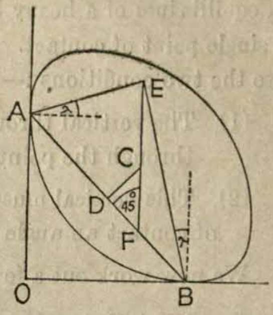


Figure shows the section of the cylinder by a vertical plane through the c. g. of the cylinder and perp. to the rough vertical plane given. Let C be the centre of the elliptic section which is the c. g. of the cylinder and A, B be the points of contact with vertical and horizontal planes and let λ be the angle of friction so that $\tan\lambda=\mu$. Draw AE, BE making angle λ with the normals at A and B. Resultant reactions at A and B act along AE, BE and they balance the weight of the cylinder ; $\therefore$ the vertical through C passes through E. Let this vertical cut AB at F and draw $CD \perp$ on AB. From symmetry AB is $\parallel$ to major axis. $\therefore$ $\angle EFA=45^\circ$ and AD, CD are the abscissa and ordinate of A referred to principal axes of the elliptic section. Let A be $(a\cos\theta, b\sin\theta)$. As tangent at A makes an angle of 45° with the major axis, we have

$$\frac{b}{a}\cot\theta=\tan 45^\circ \quad \textit{or,} \quad \cot\theta=\frac{a}{b} \quad \cdots \quad (1)$$

$$\therefore \quad AB=2a\cos\theta,\ DF=CD=b\sin\theta.$$

In $\triangle AEB$, $\angle AEF=90^\circ-\lambda$, $\angle BEF=\lambda$, $\angle EFA=45^\circ$,

$AF=a\cos\theta+b\sin\theta$ and $FB=a\cos\theta-b\sin\theta$.

Using the well-known Trigonometrical identity, we have,

$$2a\cos\theta.\cot 45^\circ=(a\cos\theta-b\sin\theta)\cot\lambda-(a\cos\theta+b\sin\theta)\cot(90^\circ-\lambda)$$

$$\textit{or,} \quad 2a\cos\theta=\frac{a\cos\theta.\cos 2\lambda-b\sin\theta}{\sin\lambda\cos\lambda}$$

$$\textit{or,} \quad b\sin\theta=a\cos\theta(\cos 2\lambda-\sin 2\lambda)$$

Using (1), $a^2(\cos 2\lambda-\sin 2\lambda)=b^2=a^2(1-e^2)$.

$$\therefore \quad e^2=\sin 2\lambda+2\sin^2\lambda=2\sin\lambda(\sin\lambda+\cos\lambda)$$

This gives e in terms of λ, the angle of friction. To find μ in terms of e, we have, $1-e^2=\cos 2\lambda-\sin 2\lambda=\dfrac{1-\mu^2}{1+\mu^2}-\dfrac{2\mu}{1+\mu^2}$

or, $\mu^2(2-e^2)+2\mu-e^2=0$. Solving the reqd. value of μ is got.

In many problems we have to consider the possible positions of equilibrium of a heavy body resting on another rough body at a single point of contact. In solving such problems we have to use the two conditions :—

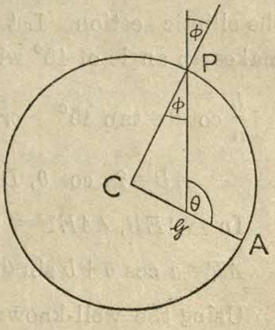
(1) The vertical through the c. g. of the body must pass through the point of contact.

(2) This vertical must not make with the normal at the point of contact an angle greater than the angle of friction.

We now work out a few sums of this type.

Ex. 4. A heavy circular hoop is hung over a rough peg. A weight equal to that of the hoop is attached to it at a given point. Find the coefficient of friction between the peg and the hoop so that the system may hang in equilibrium whatever point of the hoop is placed in contact with the peg. [C. H. 1951]

Let C be the centre of the hoop of weight ω and A the point where equal weight ω is attached. Their combined c. g. is G, the mid. point of CA. Let P be the position of the peg when equilibrium is maintained. Then GP is vertical and GP makes with normal CP an angle ϕ which is $\leqslant \lambda$. Let $\angle AGP = \theta$. For different positions of P, θ varies from 0 to π. From $\triangle CPG$, we have,



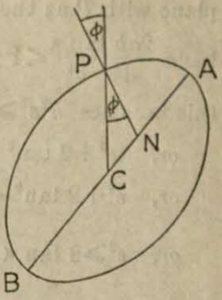
$$\frac{CP}{\sin\theta}=\frac{CG}{\sin\phi} \quad \therefore \quad \tfrac{1}{2}\sin\theta=\sin\phi\leqslant\sin\lambda.$$

$\therefore$ In order that equilibrium may hold for all values of θ where $0\leqslant\theta\leqslant\pi$, we must have $\sin\lambda\geqslant\frac{1}{2}\sin\theta$, for all such values of θ. As $\sin\theta\leqslant 1$ when $0\leqslant\theta\leqslant\pi$, the condition will hold if

$\sin\lambda\geqslant\frac{1}{2}$ *or*, $\lambda\geqslant 30^\circ$. But $\lambda<90^\circ$ $\therefore$ $\mu=\tan\lambda\geqslant\frac{1}{\sqrt{3}}$

$\therefore$ Least value of the coefficient of friction is $\frac{1}{\sqrt{3}}$.

Ex. 5. A uniform heavy elliptical wire, semi-axes a and b, is hung over a small rough peg. Show that if the wire can be in equilibrium with any point of it in contact with the peg, the coefficient of friction must not be less than $\frac{a^2-b^2}{2ab}$. [C. H. 1969]



Let C be the centre of the ellipse, CA its semi-major axis and P the position of the peg when wire is in equilibrium. Then CP is vertical and if ϕ be the angle CP makes with the normal PN at P, then we must have $\phi \leqslant \lambda$, the angle of friction. Let P be $(a \cos \theta,\ b \sin \theta)$. The slope of normal $PN = \frac{a}{b} \tan \theta$ and slope of $CP = \frac{b}{a}$ $\tan \theta$

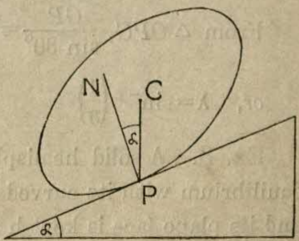
$$\therefore \quad \tan \phi = \frac{\frac{a}{b} \tan \theta - \frac{b}{a} \tan \theta}{1 + \frac{a}{b} \cdot \frac{b}{a} \tan^2 \theta} = \frac{(a^2-b^2) \tan \theta}{ab \sec^2 \theta} = \frac{a^2-b^2}{2ab} \sin 2\theta.$$

$$\therefore \quad \frac{a^2-b^2}{2ab} \sin 2\theta = \tan \phi \leqslant \tan \lambda \leqslant \mu.$$

$\therefore$ Equilibrium will hold for all positions of P if $\mu \geqslant \frac{a^2-b^2}{2ab} \sin 2\theta$, for all values of θ such that $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

As $\sin 2\theta \leqslant 1$ when $0 \leqslant \theta \leqslant \frac{\pi}{2}$, the condition will hold if $\mu \geqslant \frac{a^2-b^2}{2ab}$.

Ex. 6. A perfectly rough plane is inclined at an angle α to the horizon; show that the least eccentricity of the ellipse which can rest on the plane is $\sqrt{\frac{2 \sin \alpha}{1+\sin \alpha}}$. [C. H. 1964]



Let C be the centre of the ellipse of semi-axes a, b. Let P be its point of contact with the inclined plane in position of equilibrium. Then PC is vertical and if PN be the normal at P then $\angle CPN = \alpha$. Let P be $(a \cos \theta,\ b \sin \theta)$ referred to principal axes of the ellipse, then we know (see Ex. 5 above) that $\tan CPN = \frac{a^2-b^2}{2ab} \sin 2\theta$.

$\therefore\ \tan\alpha=\dfrac{a^2-b^2}{2ab}\cdot\sin 2\theta$. This equation determines θ *i.e.* the point P such that the ellipse may rest in equilibrium on the plane with P as the point of contact. For θ to be real we must have $\dfrac{2ab\tan\alpha}{a^2-b^2}\leqslant 1$ *or,* $a^2-b^2\geqslant 2ab\tan\alpha$; since $b^2=a^2(1-e^2)$, this becomes $a^2e^2\geqslant 2a.a\sqrt{1-e^2}.\tan\alpha$ *or,* $e^4\geqslant 4(1-e^2)\tan^2\alpha$

$$\text{or, } (e^2+2\tan^2\alpha)^2\geqslant 4\tan^2\alpha(1+\tan^2\alpha)$$

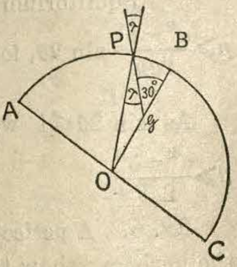
$$\text{or, } e^2+2\tan^2\alpha\geqslant\pm 2\tan\alpha\sec\alpha.$$

$$\text{or, } e^2\geqslant 2\tan\alpha(\sec\alpha-\tan\alpha)\geqslant\frac{2\sin\alpha(1-\sin\alpha)}{\cos^2\alpha}\geqslant\frac{2\sin\alpha}{1+\sin\alpha}.$$

$\therefore$ Least value of e is $\sqrt{\dfrac{2\sin\alpha}{1+\sin\alpha}}$.

Ex. 7. A uniform wire in the form of a semi-circle hangs in a vertical plane over a rough horizontal peg It is just on the point of slipping when the line joining its ends makes an angle 30° with the horizontal. Find the angle of friction.

[C. H. 1965, 1961]



Let ABC be the semi-circular wire of centre O and radius a. The c. g. of the wire is G on radius $OB\perp$ to AC where

$$OG=a\cdot\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}=\frac{2a}{\pi}.$$

Let wire be in limiting equilibrium when peg is at P and AC makes an angle 30° with horizontal. Then PG is vertical and $\angle OPG=\lambda$, the angle of friction. $\therefore\ \angle PGB=30^\circ$.

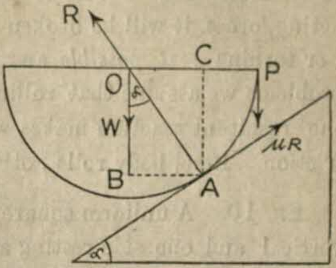
From $\triangle OPG$, $\dfrac{OP}{\sin 30^\circ}=\dfrac{OG}{\sin\lambda}$ $\therefore\ \sin\lambda=\frac{1}{2}.\dfrac{2a}{\pi}.\dfrac{1}{a}=\dfrac{1}{\pi}$

or, $\lambda=\sin^{-1}\left(\dfrac{1}{\pi}\right)$.

Ex. 8. A solid hemisphere of weight W rests in limiting equilibrium with its curved surface on a rough inclined plane, and its plane face is kept horizontal by a weight P attached to a point in the rim. Prove that the coefficient of friction is

$P/\sqrt{W(2P+W)}$, [C. H. 1963]

Let α=angle of inclination of plane, R and μR the normal reaction and the friction at A, the point of contact. Draw AB, AC horizontal and vertical through A. The hemisphere is at rest under the forces W at O, wt. P at P and R, μR at A. Resolving along and perp. to the plane and taking moments about A, we get,



$(W+P)\sin\alpha=\mu R$ ... (1), $(W+P)\cos\alpha=R$ ... (2)

and $W.AB=P.PC$ *or*, $Wa\sin\alpha=P.(a-a\sin\alpha)$

$$\therefore\ \sin\alpha=\frac{P}{W+P}.$$

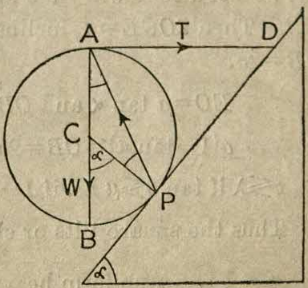
From (1) and (2), $\mu=\tan\alpha$

$$\therefore\ \mu=\frac{P}{\sqrt{(W+P)^2-P^2}}=\frac{P}{\sqrt{W(2P+W)}}.$$

When a body is at rest on a rough body at a single point of contact being acted upon by several forces, their resultant must pass through the point of contact and make with the normal at that point an angle not greater than the angle of friction.

Ex. 9. A sphere rests on a rough inclined plane, the coefficient of friction being μ, and is supported by a horizontal string attached to the highest point of the sphere and to the plane. Find the inclination of the plane when the sphere is on the point of slipping. [C. H. 1964]

Let C be the centre of the sphere resting on plane at P. A its highest point so that ACB is the vertical diameter, AD the horizontal string. The sphere is in limiting equilibrium. The wt. W of the sphere along AC and tension T along AD intersect at A. $\therefore$ The resultant reaction at P must act along PA and $\angle CPA=\lambda$, the angle of friction. Now $\angle BCP=\angle CPA+\angle CAP=\lambda+\lambda=2\lambda$.

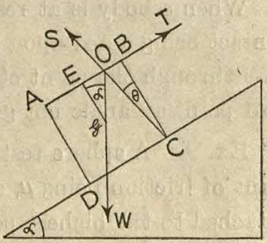


$\therefore$ inclination of plane$=\alpha=\angle BCP=2\lambda=2\tan^{-1}\mu$.

Initial Motion : When equilibrium of a body resting on a rough surface is about to be broken due to gradual change in acting forces, it will be broken at a point of contact by rolling (or turning), *if possible* and not by sliding. So in solving such problems we assume that rolling takes place and find whether the resultant reaction makes with the normal an angle < angle of friction. If so, body rolls ; otherwise it slides.

Ex. 10. A uniform square lamina is placed with its plane vertical and one side resting along the line of greatest slope on a rough inclined plane, the coefficient of friction being μ. A string is attached to the upper corner and pulled in the direction parallel to the line of greatest slope up the plane ; show that if the tension be gradually increased, the square will slide or tilt according as the angle of inclination of the plane to the horizontal is less or greater than $\tan^{-1}(1-2\mu)$. [C. H. 1966]

Let $ABCD$ be the square, G its c.g. ; and B the point where string is attached. Equilibrium may be broken either by the square sliding up the plane or by its tilting about end C. According to the rule : *If a body can roll, it will roll,* we assume that the square is about to tilt at C. Then the resultant reaction acts at C and if O is the point where OG, the line of action of the weight W meets AB, the line of action of the tension T, then the resultant reaction at C passes through O. Let $\angle BCO=\theta$; then if $\theta<\lambda$, the square will tilt ; otherwise it will slide up the plane. Now let GE be $\perp$ on AB. Then $\angle OGE=\alpha$, inclination of the plane, and $EB=a$ where $AB=2a$.



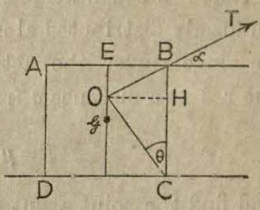
$\therefore$ $EO=a\tan\alpha$ and $OB=a-a\tan\alpha$.

$\therefore$ $a(1-\tan\alpha)=OB=2a\tan\theta$.

$\theta\lesseqgtr\lambda$ if $\tan\theta\lesseqgtr\mu$ *i.e.* if $1-\tan\alpha\lesseqgtr2\mu$ or $\alpha\gtreqless\tan^{-1}(1-2\mu)$.

Thus the square tilts or slides acc. as $\alpha\gtreqless\tan^{-1}(1-2\mu)$.

Ex. 11. A uniform heavy cube rests on a rough horizontal plane and a string is attached to the middle point of one of its upper edges. The string lies in the vertical plane passing through the centre of the cube and bisecting the upper edge and makes an angle α with the horizontal. If the string is pulled with an increasing force find the nature of initial motion of the cube.



Let $ABCD$ be the vertical section in which the string lies. Let the vertical through the c.g. G meet the string produced at O and AB at E. Then angle EBO is α. Two types of motion are possible :—

(1) The cube may slide in the direction DC ;

(2) The cube may turn about edge C.

The second will happen, if possible. So we assume that the cube turns about edge C. Then the resultant reaction at C acts along CO.

Let $\angle BCO=\theta$; if $\theta<\lambda$, the cube will turn about edge C.

To calculate θ, let $AB=2a$, then $EB=a$.

$\therefore\ HB=OE=a\tan\alpha \quad \therefore\ CH=2a-a\tan\alpha.$

$\therefore\ a=OH=CH\tan\theta=a(2-\tan\alpha).\tan\theta.$

$\therefore\ \theta\lesseqgtr\lambda$ if $1\lesseqgtr(2-\tan\alpha)\tan\lambda$ or $\mu\gtreqless\dfrac{1}{2-\tan\alpha}.$

Hence if $\mu>\dfrac{1}{2-\tan\alpha}$, cube will turn about C ; otherwise it will slide.

We know that if a plane lamina lying on a plane moves from one position to another it can be brought from its initial to its final position by rotation about some point through the proper angle. When displacement is very small this point is called the *instantaneous centre of rotation*. There is a kind of problem in which we do not know how the body moves when equilibrium is broken and so the directions of the frictional forces are unknown. To solve such problems we assume some point as the *instantaneous centre* of rotation of the body. Then the direction of motion of each point of the body and consequently those of the forces of friction become known. The two equations of resolution and the equation of moment then determine the position of the centre of rotation and the condition of motion. The methods of procedure to be adopted to solve such problems are illustrated in the solution of the following examples.

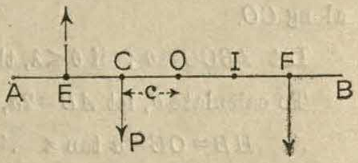
Ex. 12. A uniform rod of length $2a$ and weight W rests on a rough horizontal plane (coeff. of friction μ), its weight being uniformly distributed along its length. If the rod is just about to move under the action of a force P applied perpendicular to the rod at a distance c from the centre, show that

$$P=\mu W\cdot\frac{\sqrt{a^2+c^2}-c}{a},$$

and find the point about which the rod will begin to turn.

[C. H. 1965]

Let AB be the rod and O its centre, C the point where the force P is applied ; then $OC=c$, $AO=OB=a$. Every small element dx of the rod whose weight is wdx, where $w=$wt. of rod per unit length, exerts a pressure on the table equal to wdx and therefore the limiting friction at it is μwdx. Let I be the instantaneous centre of rotation. Then the friction at each element acts perpendicular to the line joining it to I and all these frictions and the applied force P at C are in equilibrium. If each force of friction and the force P be rotated about their points of application in the same sense through 90°, they will form a couple. But then all the frictions act towards I and so must have a resultant parallel to the rod AB along which P now acts. This is impossible unless I lies on AB. Again I cannot be to the left of A or to the right of B for then all the frictions will act ⊥ to AB in same sense and have a resultant at O and so cannot balance P at C. ∴ I is some point of AB. Let $AI=y$. The frictions at elements of AI act ⊥ to AI in direction opposite to P and those at elements of IB act ⊥ to IB in same sense as P. (as at E and F in fig.). The resultant frictions on AI and IB are μwy and $\mu w(2a-y)$ acting at middle points of AI and IB respectively. Resolving perp. to AB and taking moments about end A, we have,



$$\mu\omega y=P+\mu\omega(2a-y) \quad or, \quad P=2\mu\omega(y-a)$$

$$\text{and } \mu\omega y.\tfrac{y}{2}=P.(a-c)+\mu\omega(2a-y).\tfrac{1}{2}(y+2a)$$

$\therefore \quad y^2=4(a-c)(y-a)+(4a^2-y^2)$

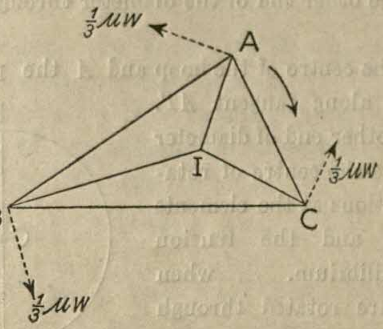
or, $\quad y^2-2(a-c)y-2ac=0 \qquad \therefore \quad y=(a-c)+\sqrt{a^2+c^2}$

$\therefore \quad P=2\mu\cdot\frac{W}{2a}\cdot(\sqrt{a^2+c^2}-c)=\frac{\mu W}{a}(\sqrt{a^2+c^2}-c)$, and rod turns about I where $AI=(a-c)+\sqrt{a^2+c^2}$.

Ex. 13. A triangular table with a point of support at each corner A, B, C is placed on a rough horizontal floor. Find the least couple which will move the table.

(Full discussion on various cases required.) [C. H. 1945]

Let ABC be the triangular table and W its weight. Then normal pressures at A, B, C are all $\frac{1}{3}W$. $\therefore$ the limiting forces of friction at each is $\frac{1}{3}\mu W$ where μ=coeff. of friction.



Two cases are to be considered separately acc. as I, the instantaneous centre of rotation is one of A, B, C or different from them.

Case I. Let I be different from A, B, C.

The frictions balance a couple and are, therefore, equivalent to a couple. So when they are rotated through 90° each, they will form a couple or be in equilibrium. As now they act along AI, BI, CI, they cannot form a couple. $\therefore$ They are in equilibrium. Hence I must be some point within $\triangle ABC$. As the frictions are all equal, by Lami's theorem, we have

$$\angle BIC=\angle CIA=\angle AIB=120^\circ$$

$\therefore$ If none of the angles A, B, C is $>120^\circ$, I is within $\triangle ABC$ and the reqd. least couple is $\frac{1}{3}\mu W(IA+IB+IC)$.

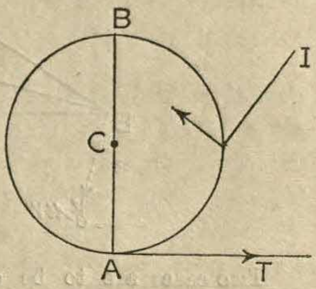
Case II. Let I coincide with one of A, B, C, say A.

Since A does not move, the friction at A is not $\frac{1}{3}\mu W$, but at B and C, the limiting frictions are each $\frac{1}{3}\mu W$. Rotating the frictions as before, we find that the friction at A must balance the resultant of the frictions $\frac{1}{3}\mu W$, $\frac{1}{3}\mu W$ acting along BA and CA. The resultant is $2.\frac{1}{3}\mu W \cos\frac{A}{2}$; this will be $<\frac{1}{3}\mu W$, only if $\frac{A}{2}>60°$ *or,* $A>120°$. $\therefore$ I cannot coincide with A unless A is greater than 120°. Hence the table cannot turn about a corner unless the angle at that corner is greater than 120°. If it turns about A, then the required least couple is $\frac{1}{3}\mu W(BA+CA)$.

Ex. 14. A hoop is laid upon a rough horizontal plane, and a string fastened to it at any point is pulled in the direction of the tangent line at the point. Prove that the hoop will begin to move about the other end of the diameter through the point.

[M. T.]

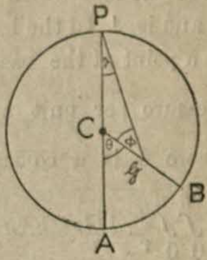
Let C be the centre of the hoop and A the point where the tension T acts along tangent AT. Let B be the other end of diameter ACB. Let I be the centre of rotation. The frictions at the elements of the hoop and the tension T are in equilibrium. $\therefore$ when the frictions are rotated through 90°, they act towards I and must have a resultant parallel to AB to form a couple with T rotated. This is impossible unless I is on AB. For otherwise the resultant, which from symmetry acts along CI, has a component ⊥ to AB. $\therefore$ I lies on AB. Now, if we consider the sum of the moments of the frictions about A, the sum must be zero. This condition is satisfied if I is at B, for then each force of friction passes through A. Hence I coincides with B and the hoop turns about B, the other end of the diameter through A.



Miscellaneous Problems :

1. A hoop of mass M hangs from a rough peg in a vertical plane. An insect of mass m starts from the lowest point of the hoop and crawls slowly upwards. Prove that the insect can reach the peg if $\sin\lambda > \frac{m}{M+m}$ but that the hoop will slip on the peg when the insect has traversed an arcual distance $a\left[\lambda+\sin^{-1}\left(\frac{M+m}{m}\sin\lambda\right)\right]$ if $\sin\lambda < \frac{m}{M+m}$. λ is the angle of friction between the hoop and the peg and a is the radius of the hoop.

Let C be the centre of the hoop and PCA the diameter through peg P; then A is the lowest point from where the insect starts. Suppose when the insect has crawled upto B, the hoop is about to slip at P. Join CB and let G be the c. g. of hoop and insect. Then $CG = \frac{m}{m+M} \cdot a$. The resultant weight of hoop and insect acts at G and is balanced by the resultant reaction at P.



$\therefore$ When insect is at B, PG becomes vertical and resultant reaction at P acts along GP, and so makes with CP, the normal at P, the angle CPG. $\therefore$ $\angle CPG = \lambda$.

Let $\angle CGP = \phi$ and $\angle ACB = \theta$

$$\therefore\ \theta = \lambda + \phi \text{ and } \frac{\sin\phi}{\sin\lambda} = \frac{CP}{CG} = \frac{m+M}{m} \quad \therefore\ \sin\phi = \frac{m+M}{m}\sin\lambda$$

$$\therefore\ \theta = \lambda + \sin^{-1}\left(\frac{m+M}{m}\sin\lambda\right). \text{ If } \frac{m+M}{m}\sin\lambda < 1$$

i.e., $\sin\lambda < \frac{m}{m+M}$ then θ is real and hoop slips at P when

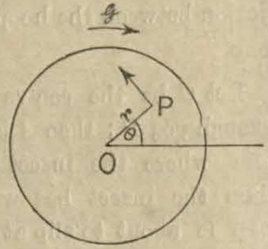
insect reaches B where arc $AB = a\theta = a\left[\lambda + \sin^{-1}\left(\frac{m+M}{m}\sin\lambda\right)\right]$.

If $\frac{m+M}{m}\sin\lambda > 1$ *i.e.*, $\sin\lambda > \frac{m}{m+M}$, θ has no real value and there is no such point as B so that the hoop slips at P when insect reaches there. In other words, the insect reaches P without any slipping taking place.

2. A solid cylinder of weight ω and radius a rests with its axis vertical on a rough horizontal plane. The coefficient of friction is μ and it is assumed that the weight of the cylinder is

so distributed over the base, that the normal pressure per unit area of the base varies inversely as the distance from the centre of the base. Find the couple required to rotate the cylinder about its axis.

Figure shows the base of the cylinder which is a circle of radius a and centre O. Let G be the least couple required to rotate cylinder about its axis which passes through O and is ⊥ to the base. Let $P(r, \theta)$ be a point of the base, then normal pressure per unit area at P is $\frac{k}{r}$, where k is a constant. ∴ Total normal pressure on base



$$=\int_0^{2\pi}\int_0^{a}\frac{k}{r}\cdot r\,d\theta\,dr = k.2\pi.a \quad \therefore \quad \omega = k.2\pi a \quad or, \quad k=\frac{\omega}{2\pi a}.$$

The normal pressure on elementary area at $P=\frac{k}{r}\cdot r\,d\theta\,dr$

$$=k\,d\theta\,dr.$$

∴ The limiting friction at the element at $P=\mu k\,d\theta\,dr$ and is perpendicular to OP.

Its moment about $O=\mu kr\,d\theta\,dr$.

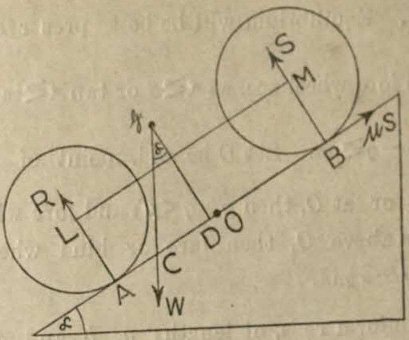
Now the frictions at the different elements of the base balance the couple G. ∴ The frictions must be equivalent to a couple of moment G. ∴ G=sum of the moments of the frictions about O

$$=\int_0^{2\pi}\int_0^{a}\mu kr\,d\theta\,dr=\mu\cdot\frac{\omega}{2\pi a}\cdot 2\pi\cdot\frac{a^2}{2}=\tfrac{1}{2}\mu\omega a.$$

Thus least couple required is $\frac{1}{2}\mu\omega a$.

3. Two equal carriage wheels whose centres are connected by a smooth bar are placed on a rough inclined plane ; determine whether the equilibrium of the system will be best preserved by locking the hind or the fore wheel. [C. H. 1941]

Let L, M be the centres of the wheels and A, B their points of contact with the plane. First suppose the hind wheel is locked and let $\measuredangle$ be the inclination of the plane when the equilibrium of the system is about to be broken. Since the joint at L is smooth, the action of the bar on the front wheel is a single force through L, and the weight of this wheel also passes through L. This wheel is in equilibrium under the action of above two forces and a third force, the resultant reaction at A. Hence the resultant reaction at A must also pass through L. If R be the resultant reaction, then R acts along AL. Let S be the normal reaction at B and as sliding takes place at B, the force of friction at B is μS up the plane. Let G be the c.g. of the system and let the vertical through G cut AB at C. Considering the equilibrium of the system and resolving along and perpendicular to the plane and taking moments about the point C, we get,



$\mu S = W \sin \measuredangle$......(1), $R + S = W \cos \measuredangle$......(2) and

$R.AC = S.BC$......(3)

Eliminating R, S, we get, $\tan \measuredangle = \dfrac{\mu.AC}{AB}$......(4).

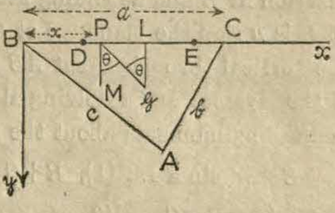
Draw $GD \perp$ on AB and let $AD = p$, $BD = q$, $GD = h$, then $p + q = AB = 2a$, say.

Then $AC = AD - CD = p - h \tan \measuredangle$ and $BC = q + h \tan \measuredangle$.

Putting these values in (4) we get, $\tan \measuredangle = \dfrac{\mu p}{2a + \mu h}$.

Next suppose the front wheel is locked and β is the corresponding inclination of the plane when system is in limiting equilibrium. In this case reaction S' at B acts along BM, and at A, which tends to slide down, there is a normal reaction R' and a friction $\mu R'$ up the plane. Resolving along and perp. to the plane and taking moments about C', the point where vertical through G now cuts AB, and proceeding as before, we find, $\tan\beta=\dfrac{\mu q}{2a-\mu h}$. Equilibrium will be best preserved by locking the hind or the fore wheel acc. as $\sphericalangle\gtrless\beta$ or $\tan\sphericalangle\gtrless\tan\beta$ or $\dfrac{\mu p}{2a+\mu h}\gtrless\dfrac{\mu q}{2a-\mu h}$ or $p-q\gtrless\mu h$. Let O be mid. point of AB. Then if D is below O or at O, then $p-q<\mu h$ and fore wheel should be locked. If D is above O, then fore or hind wheel should be locked acc. as $OD\lessgtr\frac{1}{2}\mu h$.

4. Three uniform rods, of lengths a, b and c, are rigidly connected to form a triangle ABC which is hung over a rough peg so that the side BC may rest in contact with it; find the length of the portion of the rod over which the peg may range, showing that if $\mu>\dfrac{a(a+b+c)}{b(b+c)}\operatorname{cosec} C+\tan\dfrac{C-B}{2}$, where $C>B$, the triangle will rest in any position. [M. T.]



The c.g. of the three rods is the same as that of 3 particles of wts. $(b+c)$, $(c+a)$, $(a+b)$ placed at A, B, C respectively.

Take origin at B and BC as x-axis as in fig. Then A, B, C have co-ordinates $(c\cos B,\ c\sin B)$, $(0, 0)$ and $(a, 0)$ respectively. If G be the c.g. and $GL\perp$ to BC, then

$$BL=\frac{(b+c)\,c\cos B+(a+b)a}{2(a+b+c)},\quad GL=\frac{(b+c)\,c\sin B}{2(a+b+c)}.$$

Suppose equilibrium holds when peg is at P where $BP=x$. Weight W of the rods acts at G vertically and resultant reaction R at P acts at P. $\therefore$ PG is vertical and R acts along GP. If $\angle PGL=\theta$, then R makes with PM, normal at P, an angle θ.

$\therefore\ \theta<\lambda$ or, $\tan\theta<\tan\lambda=\mu$ or $PL<\mu GL$.

When $BP<BL$, this gives $BL-x<\mu GL$ or $x>BL-\mu.GL$

When $BP>BL$, this gives $x-BL<\mu GL$ or $x<BL+\mu GL$.

Thus range of P, for equilibrium, is $BL-\mu GL<x<BL+\mu GL$

On BC take two points D and E such that $LD=LE=\mu.GL$. Then range of P is DE. If $\mu.GL>$both BL and CL, then range of P is whole of BC. As $C>B$, it can be easily shown that $BL>CL$. Hence the range of P will be the whole of BC if

$$\mu.GL>BL \quad \text{or} \quad \mu>\frac{a(a+b)+c(b+c)\cos B}{c(b+c)\sin B}$$

$$or,\ \mu>\frac{a(a+b+c)+c[(b+c)\cos B-(c\cos B+b\cos C)]}{(b+c)\,b\sin C}$$

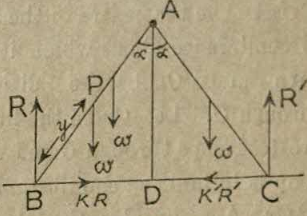
$$[\because\ a=c\cos B+b\cos C\ ;\ c\sin B=b\sin C]$$

$$>\frac{a(a+b+c)}{b(b+c)}\operatorname{cosec} C+\frac{c(\cos B-\cos C)}{(b+c)\sin C}$$

$$>\frac{a(a+b+c)}{b(b+c)}\operatorname{cosec} C+\tan\frac{C-B}{2}.$$

$$\left[\because\ \frac{c}{b+c}=\frac{\sin C}{\sin B+\sin C}\right]$$

5. Two equal uniform ladders are jointed at one end and stand with the other ends on a rough horizontal plane. A man whose weight is equal to that of one of the ladders ascends one of them. Prove that the other will slip first. If it begins to slip when he has ascended a distance x, prove that the coefficient of friction is $\left(\frac{a+x}{2a+x}\right)\tan\alpha$, a being the length of each ladder, and α the angle each makes with the vertical. [C. H. 1944]



Let AB, AC be the ladders jointed at A and let wt. of each be ω. Each makes with vertical AD an angle α. Let R, R' be the normal reactions and KR, $K'R'$ the frictions at B and C respectively when the man has ascended to P where $BP=y$.

For equilibrium of system, resolving horizontally and vertically and taking moments about C, we have

$$KR=K'R'\ldots\ldots(1)$$

$$R+R'=3\omega\ldots\ldots(2)$$

and $R.2a\sin\alpha=2\omega.a\sin\alpha+\omega(2a\sin\alpha-y\sin\alpha)$

$$or,\quad R=2\omega-\omega.\frac{y}{2a} \qquad \ldots\ldots(3)$$

$$\therefore \text{ from (2) } R'=\omega+\omega.\frac{y}{2a} \qquad \ldots\ldots(4)$$

$$\therefore\quad R \gtrless R' \text{ acc. as } 2\omega-\omega.\frac{y}{2a} \gtrless \omega+\omega.\frac{y}{2a} \text{ or } a \gtrless y.$$

Since $y<a$, $\therefore$ $R>R'$ and $\therefore$ from (1), $K<K'$.

Again, for equilibrium of AC, taking moments about A, we have,

$$R'.a \sin\alpha = K'R'\, a\cos\alpha + \omega.\tfrac{1}{2}a\sin\alpha.$$

Using (4), $\left(\omega+\omega.\dfrac{y}{2a}-\tfrac{1}{2}\omega\right)\tan\alpha = K'\left(\omega+\omega.\dfrac{y}{2a}\right)$

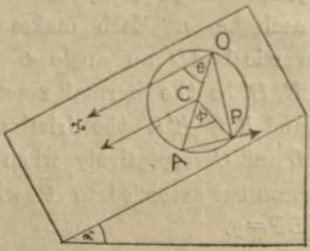
$$or,\quad K'=\left(\frac{a+y}{2a+y}\right)\tan\alpha=\left(1-\frac{a}{2a+y}\right)\tan\alpha. \quad \ldots\ldots(5)$$

$\therefore$ K' increases as y increases and will attain the value μ before K $\therefore$ AC will slip first.

As AC slips when the man has ascended a length x, therefore $K'=\mu$, when $y=x$. $\therefore$ $\mu=\left(\dfrac{a+x}{2a+x}\right)\tan\alpha$ [From (5)].

6. A uniform circular hoop lies on a rough inclined plane of inclination α and coefficient of friction μ. It is freely movable in the plane about a fixed point O on its circumference; prove that it will rest in all positions if $\tan\alpha<\dfrac{4\mu}{\pi}$.

Let C be the centre of the hoop and O the fixed point on its circumference about which it can turn and OCA the diameter through O. Let Ox be the line of greatest slope through O and when hoop is in limiting equilibrium let $\angle AOx=\theta$. Consider the element of the hoop at P where $\angle ACP=\phi$. If w = wt. per unit length of hoop, then wt. of element at $P=wds=wad\phi$, where a = radius of hoop. Normal reaction on it $=wad\phi\cos\alpha$. $\therefore$ Friction at element at $P=\mu wad\phi\cos\alpha$ acting $\perp$ to OP *i.e.* along AP. Its moment about $O=\mu wad\phi\cos\alpha.OP=\mu wad\phi\cos\alpha\, 2a\cos\dfrac{\phi}{2}$. The weight of



the hoop is $2a\pi w$ acting vertically at C. Its component along the plane is $2a\pi w \sin \sphericalangle$ acting at $C \parallel$ to Ox. The frictions at the elements of the hoop, the weight component along the plane and the reaction at O are in equilibrium.

∴ Taking moments about O, we have,

$$2a\pi w \sin \sphericalangle . a \sin \theta = \int_{-\pi}^{\pi} 2\mu w a^2 \cos \sphericalangle \cos \frac{\phi}{2} d\phi$$

$$= 2\mu w a^2 \cos \sphericalangle . 2.2$$

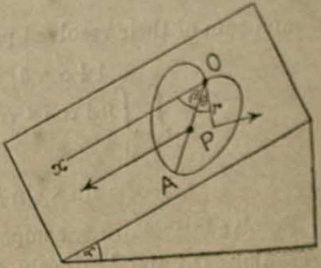
$$\therefore \quad \sin \theta = \frac{4\mu}{\pi \tan \sphericalangle}$$

If $\dfrac{4\mu}{\pi \tan \sphericalangle} > 1$ *i.e.* $\tan \sphericalangle < \dfrac{4\mu}{\pi}$, $\sin \theta$ is > 1 ∴ θ has no real value *i.e.* there is no real position of limiting equilibrium. Hence when $\tan \sphericalangle < \dfrac{4\mu}{\pi}$, the hoop rests in equilibrium in all positions.

7. A uniform disc in the shape of a cardioid lies on a rough plane inclined at ∢ to the horizon, and is capable of freely turning round a pin at its pole. When just about to slip, its axis makes an angle β with the line of greatest slope; show that $\sin \beta = \frac{4}{3}\mu \cot \sphericalangle$, where μ is the coefficient of friction.

Prove also that the direction of the action at the pin makes an angle $\tan^{-1}(\frac{1}{2} \tan \beta)$ with the axis of the cardioid.

Let O be the pole and OA the axis of the cardioid and let Ox be the line of greatest slope through O. In position of limiting equilibrium $\angle AOx = \beta$. Let equation of cardioid referred to O as pole and OA as initial line be $r = a(1 + \cos \theta)$. We know that the c.g. of the cardioid is G on OA where $OG = \frac{5}{6}a$ and area of the cardioid is $\frac{3}{2}\pi a^2$. Consider the element of area at $P(r, \theta)$. If w = wt. per unit area of the disc, wt. of the element $= wrd\theta dr$. Normal reaction on it $= wrd\theta dr \cos \sphericalangle$. Friction at the element is $\mu wrd\theta dr \cos \sphericalangle \perp$ to



OP. Its moment about $O = \mu w \cos \sphericalangle . r^2 d\theta dr$. $\therefore$ Alg. sum of the moments of the frictions at different elements of the disc

$$= \int_{-\pi}^{\pi}\int_{r=0}^{r=a(1+\cos\theta)} \mu w \cos \sphericalangle r^2 d\theta dr = \mu w \cos \sphericalangle . \frac{a^3}{3}\int_{-\pi}^{\pi}(1+\cos\theta)^3 d\theta$$

$$= \mu w \cos \sphericalangle . \frac{a^3}{3}(2\pi + 3\pi) = \frac{5\pi}{3} . \mu w \cos \sphericalangle . a^3.$$

Component of wt. of the disc along plane $= w . \frac{3}{2}\pi a^2 . \sin \sphericalangle$, acting at $G \parallel$ to Ox.

Since frictions, component of wt. of disc down the plane and the reaction at O are in equilibrium, taking moments about O, we have,

$$\frac{5\pi}{3}\mu w \cos \sphericalangle . a^3 = \frac{3w}{2}\pi a^2 \sin \sphericalangle . \frac{5}{6}a . \sin\beta,$$

or $\mu = \frac{3}{4}\tan \sphericalangle . \sin\beta$ or $\sin\beta = \frac{4}{3}\mu \cot \sphericalangle \cdots\cdots(1)$

The action at the pin balances the resultant of the frictions and the component of wt. down the plane.

To find the resultant of the frictions let them be resolved along and $\perp$ to OA. The resolved parts of the friction at the element at P are $\mu w \cos \sphericalangle r d\theta dr . \cos(90+\theta)$ along OA and $\mu w \cos \sphericalangle r d\theta dr \sin(90+\theta) \perp$ to OA.

$\therefore$ sum of their resolved parts along OA

$$= \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} -\mu w \cos \sphericalangle\, r\, d\theta dr . \sin\theta = 0$$

and sum of their resolved parts $\perp$ to OA

$$= \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} \mu w \cos \sphericalangle\, r d\theta dr \cos\theta = \mu w \cos \sphericalangle . \pi a^2$$

$$= \frac{3}{4}\pi a^2 w \sin \sphericalangle \sin\beta \text{ [from (1)]}$$

Resolved parts of wt. component along plane are $\frac{3}{2}\pi a^2 w \sin \sphericalangle \cos\beta$ along OA and $\frac{3}{2}\pi a^2 w \sin \sphericalangle \sin\beta \perp$ to OA but in opposite direction. $\therefore$ Resultant of the frictions and the wt. component along plane has components $\frac{3}{2}\pi a^2 w \sin \sphericalangle \cos\beta$ along OA and $\frac{3}{4}\pi a^2 w \sin \sphericalangle \sin\beta \perp$ to OA. Hence the resultant makes with OA

an angle $\tan^{-1}(\frac{1}{2}\tan\beta)$ and the action of the pin being equal and opposite to this resultant also makes the same angle with OA.

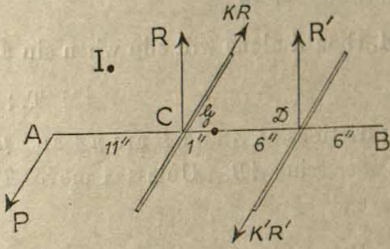
8. A uniform rod AB 24 in. in length is lying in a horizontal position upon two parallel horizontal equally rough rods and at right angles to them. The points of contact are C, D. AC is 11 in., AD 18 in. A gradually increasing force is applied at A to the rod AB parallel to the other rods. Show that the equilibrium will ultimately be broken by the rod turning about C and slipping at D. [C. H. 1949]

The initial motion of the rod is a rotation about some centre of rotation I lying in the plane of the rods. Any such rotation will mean

(1) a sliding at both C and D, when I is other than C or D

or (2) a sliding at C and a turning at D, when I is D.

or (3) a sliding at D and a turning at C, when I is C.



According to the principle that a body will roll or turn at a point of contact, if it can, in preference to sliding, we have to see first whether (2) or (3) is possible. If one of them is possible that one will happen. If none of them is possible then (1) must happen. We assume that either (2) or (3) happens. Suppose the force P is not yet sufficient to cause motion. Let R, R' be the normal reactions at C and D and let the frictions at C, D be KR, $K'R'$ where K, K' are both $<\mu$, the coeff. of friction. For equilibrium the frictions cannot be like forces so suppose they act as in fig. Let G be the c.g. of the rod AB and w its weight. Then $CG=1''$, $GD=6''$. Considering the vertical forces, which are in equilibrium, we get $R+R'=W$ and $R.1=R'.6$ $\therefore$ $R=\frac{6}{7}w$ and $R'=\frac{1}{7}w$. Considering the horizontal forces and taking moments about A, we get, $11KR=18K'R'$ $\therefore$ $66K=18K'$ $\therefore$ $K'=\frac{11}{3}K>K$.

Resolving ⊥ to AB, we get $P+K'R'=KR$

$$or,\ P=\tfrac{6}{7}W.K-\tfrac{1}{7}W\tfrac{11}{3}K=\frac{KW}{3}$$

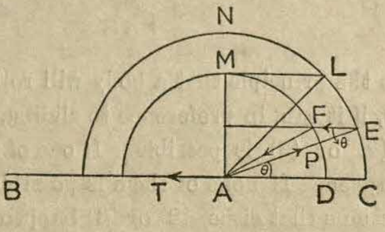
$$\therefore\ K=\frac{3P}{W} \text{ and } K'=\frac{11P}{W}.$$

Thus as P increases K and K' both increase but K' remains always $>K$ and $\therefore$ K' will first reach the value μ when K will be $<\mu$ and equilibrium will be broken by the rod sliding at D and turning at C. It will happen because it is possible.

9. Two weights, A and B, are connected by a string and placed on a horizontal table whose coefficient of friction is μ. A force P, $<\mu(A+B)$, is applied to A in the direction BA and its direction is gradually turned round an angle θ in the horizontal plane. If P be $>\mu\sqrt{(A^2+B^2)}$, show that both B and A will slip when $\cos\theta=\dfrac{\mu^2(B^2-A^2)+P^2}{2\mu BP}$, but if P be $<\mu\sqrt{(A^2+B^2)}$ and greater than μA, then A alone will slip when $\sin\theta=\dfrac{\mu A}{P}$.

[M. T.; Lucknow 1950]

The limiting frictions at A and B are μA and μB. Let T be the tension in the string AB. Unless A moves T is always zero and A moves if and only if the applied force P is $>\mu A$. We assume P is $>\mu A$; then the friction at A is always equal to μA, the limiting friction and T is not zero now. It is a passive force, with maximum value μB, and so adjusts itself that the resultant of it and μA, acting in proper direction, balances P. With centre A and radii P and μA draw the two circles CEN and DFM. Let P act along AE; draw $EF \parallel$ to AB cutting inner circle at F. Join FA. Then in $\triangle AEF$, AE represents P, FA represents μA and $\therefore$ EF *represents* T, the tension in the string and the three forces keep A in equilibrium. Now $CD=P-\mu A<\mu B$, as



$P<\mu(A+B)$. As θ increases, C moves up the circle CEN, and EF *i.e.* T increases. So long as EF is $<\mu B$, equilibrium of A is maintained and T being $<\mu B$, B has no tendency to move. But when C comes to a position where T becomes μB, then equilibrium of B is limiting and both A and B borders on motion. Let us assume $P>\mu\sqrt{A^2+B^2}$. Let L be the point on the circle CEN where the straight line drawn through it ∥ to AB touches the inner circle at M. Now $LM^2=AL^2-AM^2=P^2-\mu^2A^2>\mu^2B^2$. $\therefore$ $LM>\mu B$. $\therefore$ between C and L there must be one point, say E, where $EF=\mu B$ *i.e.* $T=\mu B$. Hence when P acts along AE both A and B are in limiting equilibrium and both of them are about to move. Let $\angle CAE=\theta$, then $\angle AEF=\theta$ and from $\triangle AEF$, we have

$$AF^2=AE^2+EF^2-2AE.EF\cos\theta$$

or, $$\mu^2A^2=P^2+\mu^2B^2-2P.\mu B\cos\theta$$

or, $$\cos\theta=\frac{\mu^2(B^2-A^2)+P^2}{2\mu PB},$$ gives the direction of P when both A and B move.

When $P<\mu\sqrt{A^2+B^2}$ but $>\mu A$, we have $LM<\mu B$. $\therefore$ when the direction of P changes continuously from AC to AL, the tension T of the string AB never reaches the value μB and $\therefore$ B never moves. But when P acts in the direction AL, the resultant of P and T represented by AL and LM acts along AM which is ⊥ to AB, and is equal to μA. $\therefore$ A is about to move being in limiting equilibrium. We have, $\sin ALM=\dfrac{AM}{AL}=\dfrac{\mu A}{P}$. Thus, in this case, when direction of P makes an angle θ with BA produced given by $\sin\theta=\dfrac{\mu A}{P}$, A alone moves.

EXAMPLES

1. A circular hoop hangs over a horizontal peg and a weight 1·5 times that of the hoop hangs tangentially from it. Show that if the hoop is about to slip on the peg the coefficient of friction is ·75.

2. A hemispherical shell rests on a rough inclined plane whose angle of friction is λ; show that the inclination of the plane base of the rim to the horizon cannot be greater than $\sin^{-1}(2\sin\lambda)$.

3. A rigid framework in the form of a rhombus, of side a and acute angle α, rests on a rough peg whose coefficient of friction is μ. Show that

the distance between the two extreme positions which the point of contact with the peg can have is $a\mu \sin \alpha$.

4. A uniform circular ring, of weight W, has a particle of weight ω attached to its circumference. Show that it can be placed over a rough peg in a position of equilibrium with any point in contact with the peg, provided that the angle of friction is not less than $\sin^{-1} \dfrac{\omega}{\omega + W}$.

5. A uniform rod ACB, length $2a$, is supported against a rough wall by a string attached to its middle point C. Show that the rod can rest with C at any point of a circular arc, whose extremities are distant a and $a \cos \lambda$ from the wall, where λ is the angle of friction.

6. A uniform ladder with its lower end on a rough ground leans against a smooth vertical wall. Prove that: (*i*) if the inclination θ of the ladder to the wall is less than the angle of friction λ, no load placed on the ladder, however great, can make it slip; (*ii*) if $\lambda < \theta < \tan^{-1} (2 \tan \lambda)$, the ladder can be made to slip by placing on it an additional load; and (*iii*) if $\theta > \tan^{-1} (2 \tan \lambda)$ the ladder will slip without an additional load.

7. A uniform heavy elliptic lamina rests with its minor axis ($2b$) vertical on a rough horizontal plane. A string is attached to the centre and is pulled horizontally in the plane of the lamina, until the major axis ($2a$) of the lamina is vertical. Show that if there is no slipping the coefficient of friction between the lamina and the horizontal plane cannot be less than $\dfrac{(a^2 - b^2)}{2ab}$.

8. Find the least coefficient of friction between a given elliptic cylinder and a particle, in order that for all positions of the cylinder in which the axis is horizontal, the particle may be capable of resting vertically above the axis. [W.]

9. On the top of a fixed rough sphere of radius r rests a thin uniform plank, and a man stands on the plank at this point. Show that he can walk slowly a distance $(n+1)r\lambda$ along the plank without its slipping off the sphere, if the weight of the plank is n times that of the man, and λ is the angle of friction between the plank and cylinder.

10. A uniform hemisphere, of radius a and weight W, rests with its spherical surface on a horizontal plane, and a rough particle, of weight W', rests on the plane surface; shew that the distance of the particle from the centre of the plane face is not greater than $\dfrac{3Wa\mu}{8W'}$, where μ is the coefficient of friction.

11. A circular disc, of radius a and weight W, is placed within a smooth sphere, of radius l, and a particle, of weight ω, is placed on the disc. If the coefficient of friction between the particle and the disc be μ, find the greatest distance from the centre of the disc at which the particle can rest.

12. A solid homogeneous hemisphere of radius a and weight W rests with its curved surface on a fixed horizontal plane, and a particle of weight W' is placed on it at a distance x from the centre; prove that, in the position

of equilibrium, the friction exerted between the particle and the hemisphere is equal to $8aW'^2(9a^2W^2+64a^2W'^2)^{-\frac{1}{2}}$. [C. H. 1959]

13. A circular disc, of radius a, whose centre of gravity is distant c from its centre, is placed on two rough pegs in a horizontal line distant $2a\sin\sphericalangle$ apart. Show that all positions will be possible positions of equilibrium, provided $a\sin\sphericalangle\sin(\lambda_1+\lambda_2)>c\sin(2\sphericalangle\mp\lambda_1\pm\lambda_2)$ where λ_1, λ_2 are the angles of friction at the two pegs.

14. A hollow cylinder with inner and outer radii a and b has one end pressed against a rough plane with a force P. Assuming that the pressure is uniformly distributed over the area, find the couple necessary to turn the cylinder about its axis while the pressure is maintained. [C. H. 1957]

15. A rod rests partly within and partly without a box in the shape of a rectangular parallelopiped, presses with one end against the rough vertical side of the box, and rests in contact with the opposite smooth edge. The weight of the box being four times that of the rod, show that, if the rod be about to slip and the box about to tumble at the same instant, the angle the rod makes with the vertical is $\frac{1}{2}\lambda+\frac{1}{2}\cos^{-1}(\frac{1}{3}\cos\lambda)$, where λ is the angle of friction. [C. H. 1948]

16. A right circular cone, of weight W and vertical angle $2\sphericalangle$, is placed with its vertex downwards and supported by a circular hole cut in a horizontal table. If μ be the coefficient of friction and b, the radius of the hole, show that the moment of the least couple about the axis that will move the cone is $\mu Wb\operatorname{cosec}\sphericalangle$. [C. H. 1950]

17. A person tries to pull out a two-handled drawer by pulling one of the handles perpendicularly to its front. Find the conditions under which the drawer will stick fast. The handles are given to be equidistant from the sides of the drawer. [C. H. 1942]

18 Prove that an ordinary drawer cannot be pushed in by a force applied to one handle until it has been pushed in a distance $2\mu c$ by forces applied in some other manner, where $2c$ is the distance between the handles and μ the coefficient of friction. [Banaras 1942]

19. Two carriage wheels A, B, of equal radii, a, but of unequal weights, P, Q, are connected by a light bar of length c attached to their centres; they are placed on a rough plane with their common plane vertical and the inclination of the rough plane is gradually increased; show that if slipping commences at the same inclination whether the front wheel, A, or the back B, is locked $\dfrac{Q-P}{Q+P}c=\mu a$.

20. Two ladders of equal lengths and of weights W, W' ($W>W'$), hinged together, form a step ladder which will just stand on a rough ground when the angle between them is $2\sphericalangle$; if μ be the coefficient of friction between the ground and the ladders prove that $\mu=(W+W')\tan\sphericalangle/(W+3W')$.

21. A uniform rod of length $2a$ rests on a rough plane inclined to the horizon at an angle $\sphericalangle$, greater than the angle of friction λ, and turn about a

point in it distant b from the c. g. of the rod. Prove that the greatest possible inclination of the rod to the line of greatest slope is

$$\sin^{-1}\left(\frac{a^2+b^2}{2ab}\cdot\frac{\tan\lambda}{\tan\alpha}\right).$$

22. A flat circular heavy disc lies on a rough inclined plane and can turn about a pin in its circumference; show that it will rest in any position if $32\mu > 9\pi \tan i$, where i is the inclination of the plane to the horizon. The weight is supposed to be equally distributed over its area.

23. A sphere rests on a rough horizontal plane and its highest point is joined to a peg fixed in the plane by a tight cord parallel to the plane. Show that, if the plane be gradually tilted about a line in it perpendicular to the direction of the cord, the sphere will not slip until the inclination becomes equal to $\tan^{-1} 2\mu$, where μ is the coefficient of friction. [C. H. 1946, M. T.]

24. A uniform hemisphere, placed with its base resting on a rough inclined plane is on the point of sliding down. A light string attached to the point of the hemisphere farthest from the plane, is then pulled in a direction parallel to and directly up the plane. If the tension of the string be gradually increased until the sphere begins to move, it will slide or tilt according as $13 \tan\alpha \lesseqgtr 8$, where α is the inclination of the plane to the horizon. [London]

25. A heavy uniform triangular plate ABC rests with its plane vertical and the side BC on a rough horizontal plane. A string attached to A and parallel to BC is pulled horizontally in the direction BC with a gradually increasing force. Show that the plate will tilt about C if $\mu > \dfrac{a(a+b\cos C)}{6\Delta}$, where Δ = area of plate.

26. A solid triangular prism is placed with its axis horizontal on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the prism. [C. H. 1943]

27. $ABCD$ is a uniform square plate which rests with its plane vertical and the side BC on a rough inclined plane of angle α. A is the highest point of the plate and is at the same level as the top T of the inclined plane. Prove that when a gradually increasing force is applied at A in the direction AT, the plate will begin to turn about B before it slips if $(1+\tan\alpha) < 2\tan(\alpha+\lambda)$, where λ is the angle of friction. [Madras 1933]

28. A uniform square lamina $ABCD$ has weight W, and the length of a side is $2a$. The diagonal AC is vertical and the sides BC, CD rest on two pegs at the same level and at distance $\sqrt{2}a$ apart, the coefficient of friction being $\frac{3}{4}$. A small horizontal force F in the plane of the lamina is applied at A and gradually increased. Determine how and for what value of F, the equilibrium is broken. [M. T.]

29. Two equally rough planes, both inclined at an acute angle α to the horizontal, intersect in a horizontal line; a uniform circular cylinder of radius a and weight W rests in the trough so formed. A slowly increasing couple is applied about the axis of the cylinder. Show that it will begin to roll up one of the planes if the coefficient of friction μ is greater than $\tan\alpha$.

If $\mu < \tan \alpha$, show that the least couple required to rotate the cylinder about its axis is $\mu Wa \sec \alpha/(\mu^2+1)$. [M. T.]

30. A right circular cylinder, of radius a, whose centre of gravity is at a distance c from the axis, is placed in the angle between a horizontal and a vertical plane so that its axis is horizontal. If the planes be of equal roughness, find in what positions it will roll and in what positions it will slip.

31. A semicircular disc rests in a vertical plane with its curved edge on a rough horizontal and an equally rough vertical plane, the coefficient of friction being μ. Show that the greatest angle that the bounding diameter can make with the horizontal plane is $\sin^{-1}\left(\frac{3\pi}{4}\cdot\frac{\mu+\mu^2}{1+\mu^2}\right)$.

32. A uniform rough beam AB lies horizontally upon two others at points A and C; show that the least horizontal force applied at B in a direction perpendicular to BA, which is able to move the beam, is the lesser of the two forces $\frac{1}{2}\mu W$ and $\mu W\frac{b-a}{2a-b}$, where AB is $2a$, AC is b, W is the weight of the beam, and μ the coefficient of friction.

33. A cube of weight W and edge $2a$ rests on a rough horizontal plane. A ladder of weight W' and length $2l$ rests symmetrically against a smooth vertical face of the cube, making an angle α with the plane where $2l \sin \alpha > a$. If the cube were fixed, a man of weight ω could ascend the ladder to a vertical height h before it slips. Prove that, if he attempts to ascend, the cube will turn round its further edge before he reaches this height, if $\omega h > (Wa - W'l \cos \alpha) \tan \alpha$, the coefficient of friction between cube and ground being $> \frac{a}{2l \sin \alpha}$.

34. A uniform rod AB of length $2a$ and weight W rests on a rough plane, its weight being uniformly distributed along its length. If the coefficient of friction be μ, and the rod is just about to move under the action of a force P applied at B perpendicular to the rod, find P and the point about which the rod will begin to turn. [C. H. 1968]

If the pressure of an element of the rod on the plane vary as its distance from the end B of the rod, and P, Q be the forces applied at B and A respectively which will just move the rod, prove that $P : Q = 2(\sqrt[3]{2} - 1)$.

35. Two uniform rods AB, BC, of weights W and W', are smoothly jointed at B and are placed so as to be in a straight line on a rough horizontal table; the end A is acted on by a gradually increasing force P in a direction perpendicular to the rods. Find how the equilibrium is broken. [Allahabad 1942]

36. Two heavy particles A, B, placed on a rough table, are connected by a string. The particle A is acted on by a force P in a given direction AC making with BA produced an angle β less than a right angle. As P is gradually increased from zero, show that A will move first or they will both move together according as $A \cot \beta$ is less or greater than B.

(A, B representing the wts. of particles A and B).

37. Four equal heavy particles A, B, C, D are connected together by weightless rods so as to form a rigid quadrilateral and placed on a rough horizontal plane. Supposing the pressures at the four particles are equal, find the least couple which will move the system.

38. A heavy uniform circular plate lies on a rough horizontal table ; to a given point on the circumference is attached a string which is pulled horizontally by a gradually increasing force. Show that the point about which the plate will begin to rotate lies on the diameter perpendicular to the string.

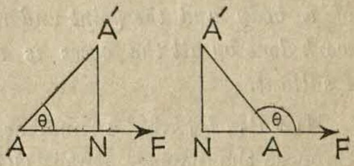
39. A heavy elliptic disc, placed on a rough table and acted on by a horizontal force F, begins to turn about a focus. If its weight be uniformly distributed over the area, prove that the force F must act along an ordinate at a distance $\frac{2a}{3} \cdot \frac{1-e^2}{e}$ from the centre.

40 A uniform semi-circular wire of weight W, rests with its plane horizontal on a rough table (coefficient of friction $= \mu$). AB is the diameter joining its ends, and C is the mid-point of the arc ; a string tied to C is pulled gently in the direction CA, and the tension increased until the wire begins to move. Show that the tension is then $2\sqrt{2}\mu W/\pi$. [C. H. 1950]

CHAPTER III

THE PRINCIPLE OF VIRTUAL WORK

3·1. In statics, we always have a body or a system of bodies in equilibrium under the action of given forces. The general method of solving problems is to write down the equations of conditions of equilibrium i.e. equations of resolution and equations of moments. But a great disadvantage is that such equations contain unknown forces of reactions, thrusts, tensions etc. which are to be eliminated. The principle of virtual work affords us a method of writing down equations involving no such unknown forces or involving only such unknown forces which we want to find out. It is a powerful method and may be used with great advantage.



3·2. If a force F act at a point A of a body and the point of application A undergoes a very small displacement AA', then *the product $F.AA'.\cos\theta$, where θ is* the angle between the direction of the force F and the direction of the displacement AA', *is called the work done by the force F in the displacement AA'.* Let $A'N$ be drawn perpendicular to the line of action of F. Then AN is the orthogonal projection of AA' on the line of action of F and is equal to $AA'\cos\theta$ and the work done is $F.AN$. So the work done by a force in any displacement is *equal to the product of the force and the orthogonal projection of the displacement on the direction of the force.* AN is also called *the displacement in the direction of the force*; it is to be considered positive or negative according as $\vec{AN}$ and $\vec{F}$ have same or opposite directions. Again, $F.AA'\cos\theta = AA'.F\cos\theta$. So the work done in any displacement is also *equal to the product of the displacement and the resolved part of the force in the direction of the displacement.* No work is done when the displacement is perpendicular to the force. Work done is a scalar quantity and not a vector like force.

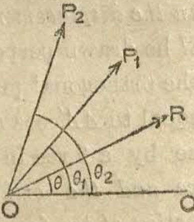
It may be noted that the work done by the force $\vec{F}$ in a displacement $\vec{AA'}$ is the scalar product $\vec{F}.\vec{AA'}$.

3·3. When the point of application A of the force F does not actually move to A' but is *imagined or conceived* to move to A', then the displacement AA' is called a *Virtual displacement* of the point A and the corresponding work done by the force F, on the assumption that its point of application has undergone the displacement AA', is called the *Virtual work done by the force F in the virtual displacement* AA'. It is of course measured by $F.AA' \cos \theta$ or $F.AN$ or $AA' \times$ resolved part of F in the direction of AA'.

In statics bodies are at rest under the action of forces and so do not actually move. All displacements of the points of application of the forces are therefore hypothetical *i.e.* imagined ones and the works calculated are the works which would be done if the displacements were actually made. That is why the displacements and the works done are called *virtual.*

3·4. Theorem. *When any number of forces act at a point of a body and the point undergoes a small displacement, the total work done by all the forces is equal to the work done by their resultant.*

Let P_1, P_2...be a number of forces acting at a point O. Suppose the displacement is OO'. Let θ_1, θ_2,...be the inclinations of the respective forces to OO', and let R be their resultant whose inclination to OO' is θ.



Then the algebraic sum of the works done by the forces $P_1, P_2,\ldots$ is

$$W = P_1 \cdot OO' \cos \theta_1 + P_2 \cdot OO' \cos \theta_2 + \cdots$$
$$= OO'(P_1 \cos \theta_1 + P_2 \cos \theta_2 + \cdots)$$
$$= OO' \cdot R \cos \theta$$
$$= R.OO' \cos \theta$$

= work done by the resultant force R.

If X, Y, Z are the resolved parts parallel to the axes of co-ordinates of a force P which acts on a particle at (x, y, z) and the particle gets a small displacement whose components parallel to the axes are dx, dy, dz, then the work done dW is

$$Xdx + Ydy + Zdz.$$

In above theorem the forces are not necessarily coplanar.

3·5. Principle of virtual work : A single particle.

From the theorem in the previous article, it follows that if a system of forces (not necessarily coplanar) acts on a particle and the particle undergoes any small virtual displacement, the total virtual work done by the forces is equal to the virtual work done by their resultant. Hence if the system is in equilibrium, their resultant is zero, therefore the virtual work done by their resultant is zero and therefore the algebraical sum of the virtual works done by the forces is zero.

Conversely, if the algebraical sum of the virtual works done by the forces is zero for every arbitrary virtual displacement of the particle, then the system is in equilibrium. For, if possible, let the system have a resultant R. Give the particle a virtual displacement δ in the direction of R.

Then $R.\delta$ = virtual work done by R in displacement δ
= alg. sum of virtual works done by the forces in displacement δ
$=0$ (given).

But $\delta \neq 0$. $\therefore$ $R=0$.

Thus : *If a system of forces acting on a particle is in equilibrium, then the algebraical sum of the virtual works done by the forces is zero for every arbitrary virtual displacement of the particle and conversely.*

This is known as the Principle of Virtual work for forces acting on a particle.

In the principle of virtual work, the displacements are all virtual and infinitesimals i.e. very small.

3·6. Deduction of the conditions of equilibrium of a particle under coplanar forces from the principle of virtual work.

Let Ox, Oy be a rect. system of axes in the plane of the forces and let the particle be at A (x, y) and let (X_1, Y_1), (X_2, Y_2),...... (X_n, Y_n) be the components of the forces, acting on the particle, parallel to the axes. Let the particle undergo any arbitrary virtual displacement whose components parallel to the axes are δx, δy. The algebraical sum of the virtual works done by the forces is $(X_1\delta x+Y_1\delta y)+(X_2\delta x+Y_2\delta y)+\cdots\cdots$

$$=\delta x.\Sigma X_1+\delta y\,\Sigma Y_1.$$

If the system is in equilibrium, then by the principle of virtual work, the virtual work of the system is zero for every arbitrary displacement and therefore $\delta x.\Sigma X_1+\delta y.\Sigma Y_1=0$ for all arbitrary values of $\delta x, \delta y$. $\therefore$ $\Sigma X_1=0, \Sigma Y_1=0$.

Conversely, if $\Sigma X_1=0, \Sigma Y_1=0$, we have $\delta x.\Sigma X_1+\delta y.\Sigma Y_1=0$ for all arbitrary values of δx, δy *i.e.*, the sum of the virtual works done by the forces is zero for every arbitrary virtual displacement of the particle and, therefore, the system is in equilibrium. Hence the necessary and sufficient conditions for equilibrium of the particle are $\Sigma X_1=0, \Sigma Y_1=0$ *i.e.* the sums of the resolved parts of the forces in any two perpendicular directions are separately zero.

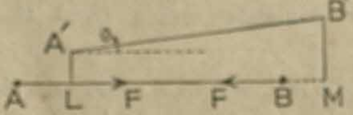
37. The principle of virtual work has been proved for a single particle. We shall now prove it for a free rigid body on the following assumptions :

(1) That a rigid body is made up of particles held together by internal forces, the distance between any two particles being invariable ;

(2) That the internal forces are of the nature of actions and reactions obeying Newton's third law i.e. mutual action and reaction between any two particles are equal and opposite, acting along the straight line joining the two particles ;

(3) That virtual displacements given to the body are infinitesimals of the first order and work done by a force is considered as zero when it is an infinitesimal of order higher than the first.

We shall first find *the virtual work done by the mutual action and reaction between two particles.*



Let A, B be the two particles and F, F, the action and reaction between them ; the action F on A acts along AB and the reaction F on B acts along BA. Due to the small virtual displacement given, let A', B' be the displaced positions of A and B respectively (the displacements are quite arbitrary and not necessarily confined to a plane). Draw $A'L, B'M$ perpendiculars on AB. The total work done by the action and reaction due to the displacement

$$=F.AL-F.BM=F(AL-BM)$$

$$=F[(AL+LB)-(LB+BM)]=F(AB-LM),$$

$=F(AB-A'B'\cos\theta)$, where θ is the small angle of the first order between AB and $A'B'$;

$=F(AB-A'B')$, neglecting θ^2, since by Taylor's theorem

$$\cos\theta=1-\frac{\theta^2}{2}\epsilon,\ \text{where}\ |\epsilon|<1.$$

$$=-F\delta l,\quad \text{where } AB=l \text{ and } A'B'=l+\delta l \quad \ldots \quad (1)$$

If the distance between the particles remains unchanged in the displacement δl is zero and ∴ work done is zero. Hence *the virtual work done by the mutual action and reaction between two particles of a rigid body is zero, if the displacement is an infinitesimal of the first order and small quantities of order higher than the first are neglected.*

Now we can prove the principle for a rigid body.

38. The principle of virtual work for a free rigid body.

If a free rigid body is in equilibrium under the action of a given system of external forces and the body is given any infinitesimal virtual displacement, the algebraical sum of the virtual works done by the external forces is zero.

Let a rigid body be in equilibrium under the action of given external forces. The rigid body is considered as being made up of particles held together by internal forces of action and reaction, the distance between any two particles being invariable. Each particle of the body is in equilibrium under the external forces acting on it and the internal forces which are the actions of the other particles of the body on it. If the body undergoes an infinitesimal displacement, each particle of the body does the same; and the total virtual work done by all the forces, external and internal, acting on a particle is zero, by the principle of virtual work for a particle, and this holds for every particle, and therefore, by summing, the total virtual work done by all the forces, external and internal, acting on all the separate particles of the body is zero. But the internal forces consist of pairs of forces of action and reaction between pairs of particles of the body and the virtual work done by each such pair of action and reaction is zero, as the particles on which they act undergo infinitesimal displacements and the distance between them remains unaltered. Therefore the total virtual work done by all the internal forces is zero and hence the total virtual work done by all the external forces is also zero.

Strictly speaking, the virtual work of the external forces is not zero, but an infinitesimal of the second order if the greatest displacement undergone by the body is an infinitesimal of the first order.

3·9. Now we consider the equilibrium of a system of rigid bodies under given external forces, in which the bodies are not free but are connected and may be also under certain constraints e. g. two smoothly hinged rods placed over a smooth cylinder. Each such connection between two bodies gives rise to mutual action and reaction between the two and the connection may be removed if we apply to each body the force of action or reaction acting on it. Similarly each constraint acting on a body gives rise to an action on the body and the constraint may be removed if we apply to the body the action the constraint produces. Suppose all the connections and the constraints have thus been removed by introducing the necessary forces, which we shall call the forces of constraint. Each body of the system is now made free and is in equilibrium under the external forces and the forces of constraint introduced to act on it. By the principle of virtual work for a free rigid body, the virtual work of all the forces acting on each body of the system is zero and therefore by addition, *the virtual work of all the external forces and the forces of constraint acting on all the separate bodies is zero.*

Hence it follows that *if a system of rigid bodies connected together and subject to certain constraints, be in equilibrium under the action of given external forces and if the system be given an infinitesimal virtual displacement in which the forces of constraint do no work, the algebraical sum of the virtual works done by the given external forces is zero.*

This is the Principle of Virtual Work in its most general form. A virtual displacement in which the forces of constraint do no work is said to be "*consistent with the geometrical conditions* *of the system*". If the displacement given to the system be such that certain forces of constraint do work, then the algebraical sum of the virtual works done by the external forces and such forces of constraint will be zero ; and such forces of constraint will appear in the general equation of virtual work.

3·10. Forces which do not appear in the equation of virtual work.

From the conclusions arrived at towards the end of the previous article it appears to be necessary that we should know what forces of constraint do not appear in the general equation of virtual work for the displacement conceived. We state some usually occurring cases.

(*a*) *Forces of action and reaction at a smooth joint.* For they act at the same point viz. the joint, and are equal and opposite and so the works done by them for any small displacement of the joint are equal and of opposite sign and so the total work done is zero.

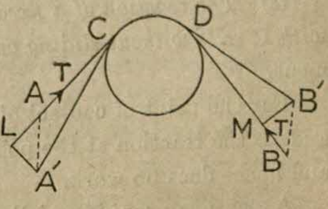
(*b*) *Forces of action and reaction between two particles when the distance between them remains invariable.* This has been proved in Art. 3·7.

(*c*) *The tensions at the ends of an inextensible string or the thrusts at the ends of a rod, when the length of the string or the length of the rod remains unaltered.*

(*i*) When the string is straight.

It follows from Art 3·7. F being replaced by T, the tension of the string.

(*ii*) When the string passes over a smooth surface. Since the surface is smooth, the tension T of the string is the same throughout its length. Let $ACDB$ be the position of the string when in equilibrium, the tensions T, T at A and B acting along AC and BD respectively ; let $A'CDB'$ be its displaced position, where $\angle ACA' = \theta$ and $\angle BDB' = \phi$ are small quantities of the first order. If $ACDB = l$ and $A'CDB' = l + \delta l$, we have $\delta l = (A'C + DB') - (AC + DB) \ldots (1)$



Draw $A'L$, $B'M$ perpendiculars on CA and DB respectively.

$$\therefore \quad CL = CA' \cos\theta = CA'\left(1 - \frac{\theta^2}{2}\eta\right), \text{ where } |\eta| < 1.$$

(see Art. 3·7)

$= CA'$, neglecting θ^2, a small quantity of second order

and, $DM = DB' \cos\phi = DB'\left(1 - \frac{\phi^2}{2}\eta'\right)$, where $|\eta'| < 1$,

$= DB'$, neglecting ϕ^2, a small quantity of secondorder.

$\therefore$ total work done by the tensions T, T acting at A and B,

$= -T.AL + T.BM = -T(CL - CA) + T(DB - DM)$

$= -T(CA' - CA - DB + DB') = -T.\delta l$, from (1).

So the work done by the tensions $= -T.\delta l$, neglecting small quantities of second order. If the length of the string remains unaltered in the displacement, then $\delta l = 0$ and, therefore work done by the tensions is zero.

The result holds when the string passes over more than one smooth surface.

(*iii*) The result in the case of a rod follows from Art. 3·7.

When the stress in the rod is a tension T, we get directly from the article :

Work done by the tensions $= -T\delta l$, and is zero if $\delta l = 0$.

When the stress in the rod is a thrust T, we have to replace F in Art 3·7 by $-T$ and we get : work done by the thrust $= T\delta l$, and is zero if $\delta l = 0$.

(*d*) *The reaction of a smooth surface when the body in contact with it slides on the surface* due to the displacement.

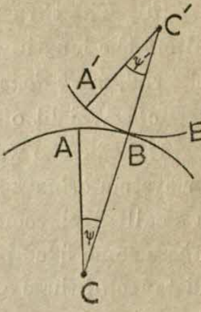
For the displacement being in the tangent plane at the point of contact is perpendicular to the reaction which is along the normal and so no work is done by the reaction.

(*e*) *The reaction of a fixed surface when the body in contact with it rolls without sliding on the surface* due to the displacement.

For the point of contact of the body is momentarily at rest and so the reaction at the point of contact has no displacement and hence does no work.

A rigorous proof is as follows :

Let the figure be the section of the surfaces by the plane through the common normal at A, their initial point of contact and containing the elementary arc AB of rolling. The lower surface is fixed, and the upper has rolled to the neighbouring position $A'BE'$, the new point of contact being B. The initial point of contact A has moved to the position A' and arc AB=arc $A'B=s$ is a small quantity of the first order. Let the normals at A and A' meet the common normal at B in the points C and C'. Then ultimately C, C' are the centres of curvatures and $CB=\rho$, $C'B=\rho'$, are the radii of curvatures of the lower and upper curves at A and A' respectively.



If $\angle ACB=\psi$, $\angle A'C'B=\psi'$ and θ is the angle through which the upper surface has rotated, we have $\theta=\psi+\psi'$, $s=\rho\psi=\rho'\psi'$ and $\therefore\ \theta=s\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)$. Since B is the instantaneous centre of rotation, we have, $\angle ABA'=\theta$ and $BA=BA'$ $\therefore$ $AA'=2AB\sin\frac{\theta}{2}=2s\cdot\frac{\theta}{2}$ $=s\theta$, neglecting small quantities of higher order.

$\therefore\ AA'=s^2\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)$, a small quantity of second order, as s is of first order and so is negligible. Hence the work done by the reaction at A is zero as the displacement of A is negligible.

(f) No work is done by a force if its point of application does not move in the displacement given. So such a force will not appear in the equation of virtual work.

It is evident that whether a force will appear in the equation of virtual work or not depends not on the force but on the displacement given.

3·11. Forces which appear in the equation of virtual work.

Of course every force which does work due to the displacement conceived will appear in the equation of virtual work, *e. g.* if any constraint is violated, the corresponding force of constraint will do work and appear in the equation. The following three deserve special mention as they very often occur in problems.

(*a*) The tensions at the ends of a string or rod when the displacement alters their lengths. T being the equal tensions and

l the length of the string or rod, *work done is* $-T\delta l$. Proved in Art. 3·10 (c).

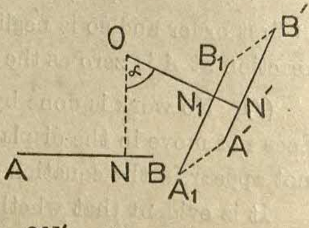
(b) The thrusts at the ends of a rod when the displacement alters the length. T being the equal thrusts and l the length of the rod, *work done is* $T\delta l$. Proved in Art. 3·10 (c).

(c) Weight of a body or of a system of bodies.

If W be the weight and y the depth of the c. g. of the body or system *below* some fixed plane, work done by the weight due to a small displacement is $W\delta y$. But if y be the height of the c. g. *above* some fixed plane, then the work done is $-W\delta y$. [For the distance of the c. g., where the weight acts, is measured positively in a direction opposite to the direction in which the weight acts.]

3·12. Lemma. *If a lamina moves on a plane from any one position to any other, it may be brought from its initial to its final position by, first a motion of rotation about any arbitrary point in the plane through the requisite angle and then a motion of translation in the proper direction and of proper magnitude.*

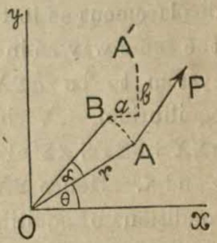
Let AB be any line in the lamina in its initial position and let $A'B'$ be the position of the same line when the lamina is in its final position. Then if AB can be brought to the position $A'B'$ by any motions of rotation and translation, the whole lamina will also be brought from its first to its last position. Let O be any arbitrarily chosen point in the plane. Draw ON, ON' $\perp^s$ on AB and $A'B'$. Rotate the lamina about O through the angle $NON' = \alpha$ so that ON falls on ON'; this brings ANB into the position $A_1N_1B_1$, and $A_1N_1B_1$ is parallel to $A'B'$ as they are both $\perp$ to ON'. Now give the lamina a motion of translation so that A_1 moves to A', then A_1B_1 moves to $A'B'$. Thus by a rotation about O through an angle α and then a translation $\overrightarrow{A_1A'}$, AB has been brought to the position $A'B'$ and so the whole lamina has been brought from its initial to its final position.



It should be noted that *every point of the lamina is brought from its initial to its final position by the same rotation* α *and the same translation* $\overrightarrow{A_1A'}$.

3·13. The principle of virtual work for any system of coplanar forces acting on a rigid body : *If a system of coplanar forces acting on rigid body be in equilibrium and if the body be given any arbitrary infinitesimal virtual displacement in the plane of the forces then the algebraical sum of the virtual works done by the forces is zero and conversely if the algebraical sum of the virtual works done by the forces be zero for every arbitrary infinitesimal displacement of the body in the plane of the forces, then the system is in equilibrium.*

Take Ox, Oy as axes of co-ordinates in the plane of the forces The most general displacement of the body, the plane of the forces remaining unaltered, is a rotation about O together with a translation. Let the body be given a rotation about O through *any small* angle α and *any small* translation whose components parallel to Ox, Oy are a, b. Let P be any force of the system acting at the point A of the body and let its components parallel to the axes be X, Y. Let (x, y) be the cartesian co-ordinates of A and (r, θ) its polar co-ordinates. Then $x=r\cos\theta$, $y=r\sin\theta$. Due to the rotation A comes to B whose polar co-ordinates are $(r, \theta+\alpha)$ $\therefore$ Its cartesian co-ordinates are $[r\cos(\theta+\alpha), r\sin(\theta+\alpha)]$



or, $[x-y\alpha, y+x\alpha]$, writing $\cos\alpha=1$ and $\sin\alpha=\alpha$, neglecting α^2, α^3 which are small quantities of second and higher order. Due to the translation B comes to the position A'.

$\therefore$ If $(x+\delta x, y+\delta y)$ be the cartesian co-ordinates of A', we have $x+\delta x=x-y\alpha+a$, $y+\delta y=y+x\alpha+b$, *or*, $\delta x=a-\alpha y$ and $\delta y=b+\alpha x$. Therefore work done by the force P

$$=X\delta x+Y\delta y=X(a-\alpha y)+Y(b+\alpha x)$$

$$=aX+bY+\alpha(xY-yX).$$

Similarly for the other forces. Hence the algebraical sum of the virtual works done by all the forces

$$=a\Sigma X+b\Sigma Y+\alpha\Sigma(xY-yX), \quad \ldots \quad (1)$$

a, b, α being the same for all the points of the body.

Let the system be in equilibrium.

Then we have $\Sigma X=0$, $\Sigma Y=0$, and $\Sigma(xY-yX)=0$.

$\therefore$ Right hand member of (1) is zero for all values of a, b and α. Hence when the body is in equilibrium, the algebraical sum of the virtual works done by the forces is zero for every arbitrary infinitesimal displacement in the plane of the forces.

Conversely, let the algebraical sum of the virtual works done by the forces be zero for every arbitrary infinitesimal displacement in the plane of the forces. Then we have $a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) = 0$ for all arbitrary small values of a, b and α.

$\therefore$ $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma(xY - yX) = 0$ and therefore the system is in equilibrium.

3·14. Assuming the truth of the principle of virtual work we can find *the conditions of equilibrium of a system of coplanar forces acting on a rigid body*. Giving the body the general displacement as in the previous article and proceeding exactly in the same way as in that article we find the virtual work of the system to be $a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX)$. As the body is in equilibrium, $\therefore$ by the principle of virtual work, we have, $a\Sigma X + b\Sigma Y + \alpha\Sigma(xY - yX) = 0$ for all arbitrary small values of a, b and α. Hence $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma(xY - yX) = 0$. These are the conditions of equilibrium.

3·15. Converse of the Principle of virtual work.

If a system of rigid bodies, connected together and subject to certain constraints, be under the action of given external forces and if the sum of the virtual works of the external forces be zero for every arbitrary infinitesimal displacement in which the forces of constraint do no work (*i.e.*, *the displacement is consistent with the geometrical conditions of the system*), *then the system is in equilibrium.*

For if the system be not in equilibrium, then the system will have a *definite* motion and each point of the system will move in a definite curve in space. By introducing certain smooth connections we make this motion the *only* motion possible for the system. If a point P of the system would now move in the direction PQ, by applying some force F at P in the direction QP we prevent the point P from moving and this prevents the whole system from moving from its initial position. The system is now in equilibrium under the given external forces, the force F, and all the forces of constraint; hence the sum of the virtual works of these forces for every arbitrary small virtual displacement consistent with the geometrical conditions is zero. Let the system be given a small virtual displacement which it can receive due to the only motion possible for it and let the point P move to Q. The work done by the external forces and the force F is zero, as the forces of constraint do no work. Also for such a displacement the work of the external forces is given to be zero. Therefore the work of F is zero, *i.e.*, $-F \times PQ = 0$. As $PQ \neq 0$, it follows that $F = 0$. Thus no force is required for preventing the motion of the system. Hence the system is in equilibrium.

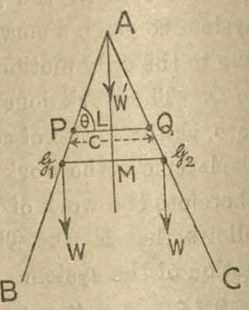
3·16. Applications of the Principle of virtual work.

A. A common application of the principle is *to find the position of equilibrium* of a body or a system of bodies under the action of given external forces. Generally the body or the system has one degree of freedom and its position is determined by a single variable, say θ. We should give the body or the system a virtual displacement which does not violate any of the constraints ; in other words, a displacement consistent with the geometrical condition of the system. The equation of virtual work will not contain any unknown reaction but will give an equation in θ, which will determine the position of equilibrium. In most of the problems, gravity is the only external force. If $\omega_1, \omega_2, \ldots\ldots$ be the wts. of the different parts of the system and $z_1, z_2, \ldots\ldots$ their depths below some fixed plane, and the system be given a small virtual displacement consistent with its geometrical condition, the equation of virtual work is $\Sigma \omega_1 \delta z_1 = 0$. This gives the required equation in θ. We shall now illustrate the method by solving a few examples.

Examples Worked

Ex. 1. Two uniform rods, each of weight W and length a, are freely jointed at A, and each passes over a smooth peg at the same level. From A a weight W' is suspended. Show that in the position of equilibrium the inclination θ of the rod to the horizon is given by $\cos^3\theta = \dfrac{c(2W + W')}{2Wa}$, c being the distance between the pegs. [C. H. 1963]

AB, AC are the rods ; their weights W, W act at their middle points G_1, G_2 ; P, Q are the pegs where PQ is horizontal



and equal to c. In position of equilibrium $\angle APQ=\theta$. Forces acting on the system formed by the two rods are (1) normal reactions at pegs P, Q, (2) weights W, W at G_1, G_2 and (3) weight W' at A. We imagine the system to be given an infinitesmial displacement in which no geometrical conditions are violated, which is symmetrical and changes θ to $\theta+\delta\theta$ (*e. g.* the joint A may be pushed vertically downwards, the rods sliding at P, Q). Work is done only by the weights W, W, W'. Since PQ is a fixed horizontal line, we shall measure the distances of A, G_1, G_2 from PQ. Height of A above $PQ=AL=\frac{c}{2}\tan\theta$. Depth of G_1 or G_2 below $PQ=LM=AM-AL$

$$=\frac{a}{2}\sin\theta-\frac{c}{2}\tan\theta.$$

$\therefore$ Equation of virtual work is,

$$2W\delta\cdot\left(\frac{a}{2}\sin\theta-\frac{c}{2}\tan\theta\right)-W'\delta\cdot\left(\frac{c}{2}\tan\theta\right)=0$$

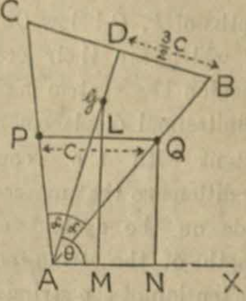
[The second term on left is given $-ve$ sign as A is above PQ and its distance is measured in the direction opposite to that in which W' acts.]

$$\therefore\quad W(a\cos\theta-c\sec^2\theta)-W'\cdot\frac{c}{2}\sec^2\theta=0$$

$$or,\quad 2W(a\cos^3\theta-c)-W'.c=0\quad\therefore\quad \cos^3\theta=\frac{c(2W+W')}{2Wa}.$$

Ex. 2. An isosceles triangular lamina with its plane vertical rests downwards between two smooth pegs in the same horizontal line. There will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical ; 2α being the vertical angle of the lamina, and the length of the base being three times the distance between the pegs. [M. T., Nagpur 1949]

Let ABC be the triangle and P, Q the pegs where $PQ=c$; Draw $AD\perp$ on BC, then $\angle BAD=\alpha$ and $BD=\frac{3c}{2}$. In position of equilibrium, let AD make an angle θ with AX, the horizontal through A. Since the pegs are smooth, the reactions at P, Q are $\perp$ to AC, AB and if the lamina be given a displacement in which sliding takes place at P, Q, no work will be done by these reactions and the weight of the lamina will be the only force which will do work. So imagine the lamina is given an infinitesimal displacement consistent with the geometrical conditions. If G be the c. g. of the lamina and GL be the $\perp$ from G on fixed line PQ and W, the weight of the lamina, the equation of virtual work is $-W.\delta(GL)=0$. Let GL meet AX at M and draw $QN \perp$ on AX. We have $AG=\frac{2}{3}AD$ $=\frac{2}{3}.\frac{3}{2}c \cot \alpha=c \cot \alpha$, $\therefore$ $GM=c \cot \alpha \sin \theta$,



$\therefore$ $GL=GM-LM=c \cot \alpha \sin \theta-QN$.

Now $QN=AQ \sin (\theta-\alpha)$ and from $\triangle APQ$, we have,

$$\frac{AQ}{\sin APQ}=\frac{PQ}{\sin PAQ} \quad or, \quad AQ=\frac{c \sin PAX}{\sin 2\alpha}=\frac{c \sin (\theta+\alpha)}{\sin 2\alpha}.$$

$$\therefore \quad \delta.\left[c \cot \alpha \sin \theta-\frac{c \sin (\theta+\alpha)}{\sin 2\alpha}\cdot \sin (\theta-\alpha)\right]=0$$

$$or, \quad c \cot \alpha \cos \theta-\frac{c}{\sin 2\alpha} \sin (\theta+\alpha+\theta-\alpha)=0$$

$$or, \quad \cot \alpha.2 \sin \alpha \cos \alpha.\cos \theta-2 \sin \theta.\cos \theta=0$$

$$or, \quad \cos \theta[\cos^2 \alpha-\sin \theta]=0$$

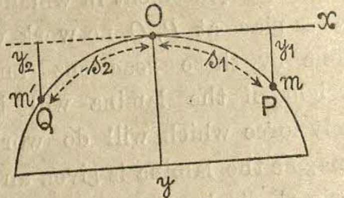
$\therefore$ Either $\cos \theta=0$ or, $\sin \theta=\cos^2 \alpha$.

When $\cos \theta=0$, $\theta=90°$.

It gives the symmetrical position of equilibrium in which AD is vertical. The other positions of equilibrium are given by $\theta=\sin^{-1}(\cos^2 \alpha)$. This gives two non-symmetrical positions of equilibrium which are symmetrically placed w. r. to the vertical position.

Ex. 3. Two particles of masses m and m' are connected by a string of length l resting on a smooth cycloid with its vertex upwards, and base horizontal. Prove that in the equilibrium position the distance of the particle m from the vertex measured along the arc is $\frac{m'l}{m+m'}$. [London]

In position of equilibrium let P, Q be the positions of the masses m and m' on the cycloid whose vertex is O and Ox, Oy are axes of co-ordinates. Let arcs OP, OQ be s_1, s_2 and let depths of P, Q below Ox be y_1, y_2 which are their ordinates. Imagine the system is given an infinitesimal displacement consistent with the geometrical conditions so that masses m, m' slide on the cycloid and the length of the string remains unaltered. ∴ No work is done by the tension of the string or the normal reactions and the equation of virtual work is



$$mgdy_1 + m'gdy_2 = 0 \qquad \dots \qquad (1)$$

Now in a cycloid we know $s^2 = 8ay$, where s is the arcual distance from the vertex and y the ordinate of a point on the cycloid.

$$\therefore \; s_1^2 = 8ay_1 \text{ and } s_2^2 = 8ay_2$$

Differentiating, $2s_1ds_1 = 8ady_1$ and $2s_2ds_2 = 8ady_2$

Further $s_1 + s_2 = l$ ∴ $ds_1 + ds_2 = 0$. ... (2)

Hence, we have, $ms_1ds_1 - m's_2ds_1 = 0$ [from (1) and (2)].

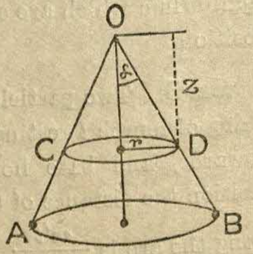
$$or, \; \frac{s_1}{m'} = \frac{s_2}{m} = \frac{l}{m'+m} \quad \therefore \; s_1 = \frac{m'l}{m+m'}.$$

Ex. 4. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1 + \frac{W}{2\pi\lambda} \cot \alpha\right).$$

[Nagpur, 1954]

When in equilibrium, let CD be the position of the elastic string on the cone and T its tension. Let r be the radius of the circle CD and z the depth of its centre below O, the vertex of the cone. Then $r = z \tan \alpha$, and the length of the string CD is $l = 2\pi r = 2\pi z \tan \alpha$.........(1) Imagine the string is given an infinitesimal displacement in which it moves vertically down, parallel to



with the cone. Since reaction at a

point of contact with the cone is normal to the surface of the cone at the point and the displacement is along the surface, no work is done by the reaction at any point ; ∴ works are done only by the tension T of the string, whose length has altered, and by the weight W of the string. ∴ Equation of virtual work is $W.\delta z - T\delta l = 0$ *or*, $W.\delta z - T2\pi \tan \sphericalangle \delta z = 0$ [from (1)]

$$or,\quad W = 2\pi \tan \sphericalangle . T. \quad \text{Now } T = \lambda.\ \frac{2\pi r - 2\pi a}{2\pi a} = \lambda.\frac{r-a}{a}.$$

$$\therefore\quad W = 2\pi \tan \sphericalangle .\ \lambda.\ \frac{(r-a)}{a} \quad or,\ r = a\left[1 + \frac{W \cot \sphericalangle}{2\pi\lambda}\right].$$

Ex. 5. Two small smooth rings of equal weights slide on a fixed elliptical wire, whose major axis is vertical, and they are connected by a string which passes over a small smooth peg at the upper focus ; show that the weights will be in equilibrium wherever they are placed. [C. H. 1953]

Let S be the focus and XY the directrix of the ellipse. Let P, Q be the positions of the rings, each of weight W, when in equilibrium, and let PM, QN be the perpendiculars from P, Q on directrix XY. Then, if e be the eccentricity of the ellipse, we know, $SP = e.PM$, $SQ = e.QN$. Give the system a small displacement without violating any constraint. Since PM, QN are depths of P, Q below fixed line XY, the eqn. of virtual work is $W\delta(PM) + W\delta(QN) = 0$ *or*, $\delta(PM) + \delta(QN) = 0$······(1)

Now length of string $= SP + SQ = e.(PM + QN)$

∴ $PM + QN =$ constant. ∴ $\delta(PM) + \delta(QN) = 0$.

Thus the condition (1) is satisfied for all positions of the particles. Hence result follows.

B. An important application of the principle is *to find unknown reactions.* A body or a system of bodies is in equilibrium under given external forces and the position of equilibrium of the body or the system is known. We have to find one or more reactions or forces of constraint of the system e.g. the tension of a string, the thrust in a rod or the action at a joint. To find any particular force of constraint, we give the system *a virtual displacement which violates the constrant that produces the constraining force* but is otherwise consistent with the geometrical conditions of the system. The equation of virtual work will contain the given external forces and this force of constraint [see Art. 3·9]. The force of constraint being the only unknown in the equation will be thus determined. In most of the problems we have to find the tension of a string or the thrust in a rod. In such cases we should give the system a displacement which *alters* (either increases or decreases) *the length of the string or the rod, without violating any other constraint.*

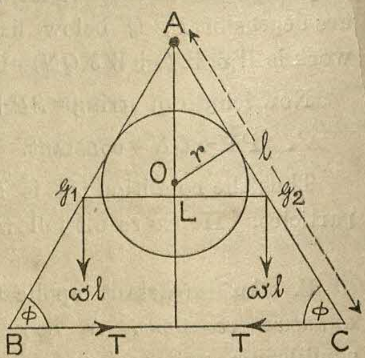
Some problems are worked out to illustrate the methods.

(A) A framework resting on pegs.

Ex. 1. Two equal rods, each of weight ωl and length l, are hinged together and placed symmetrically over a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string. [C. H. 1968]

If the string is slack, show that ϕ satisfies the equation $\tan^3\phi+\tan\phi=\dfrac{l}{2r}$. [C. H. 1963, 1956]

Let AB, AC be the rods of length l and weight ωl, and O the centre of the cylinder of radius r. Let T be the tension of the string joining B and C. In position of equilibrium, $\angle ACB=\angle ABC=\phi$. We have $AB=AC=l$; the weights ωl, ωl of the rods act at G_1, G_2, the mid points of rods AB, AC. Forces acting on the rods are (1) the actions of the smooth cylinder at the points of contact perpendicular to AB and AC; (2) the weights ωl, ωl at G_1, G_2 and (3) the tension T of string BC. Imagine the system to be given an infinitesimal displacement in which A moves vertically downwards, the rods slide over the cylinder, and the length of the string BC is altered. The actions at the points of contact of the rods with the cylinder do no work; work is done only by the weights of the rods and the tension T.



∴ Equation of virtual work is $2.\omega l\delta.(OL) - T\delta.(BC) = 0$ ···(1)

where OL = depth of G_1 *or* G_2 below fixed point O,

$$= AL - AO = AG_1 \sin\phi - r \operatorname{cosec}(90^\circ - \phi),$$

$$= \frac{l}{2}\sin\phi - r\sec\phi.$$

and $BC = 2l\cos\phi$.

∴ From (1) we get, $2\omega l.\left[\frac{l}{2}\cos\phi - r\sec\phi\tan\phi\right] - T.(-2l\sin\phi) = 0$

$$\therefore \quad T = \frac{\omega}{2}\left[2r\sec^2\phi - l\cot\phi\right].$$

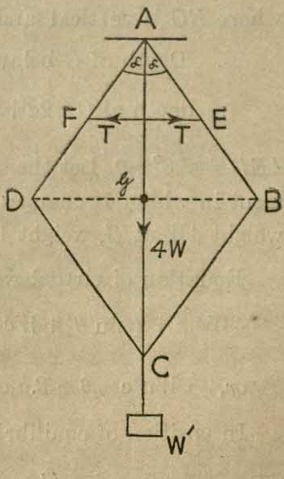
If the string is slack $T = 0$ ∴ $2r\sec^2\phi - l\cot\phi = 0$

or, $2r(1+\tan^2\phi) = l\cot\phi$ *or*, $\tan\phi + \tan^3\phi = \frac{l}{2r}$.

(B) Rods forming a rhombus.

Ex. 2. Four equal rods each of weight W form a rhombus $ABCD$ with smooth hinges at the joints The frame is suspended by the end A and a weight W' is attached to C. A stiffening rod of negligible weight joins the middle points of AB, AD keeping these inclined at an angle α to AC. Show that the thrust in the stiffening rod is $(4W + 2W')\tan\alpha$. [C. H. 1960 ; 1950]

Let T be the thrust in the stiffening rod EF joining E, F, the mid points of rods AB and AD.



Let $2a$ = length of each rod.

∴ $EF = 2a\sin\alpha$ where

$\alpha = \angle BAC = \angle DAC$;

$AC = 4a\cos\alpha$.

The weight of the 4 rods, $4W$, acts at G, their c. g, where

$AG = 2a\cos\alpha$.

Imagine the system to be given an infinitesimal symmetrical displacement in which C moves vertically downwards and no geometrical conditions are violated except that the length of rod EF is altered thus causing the thrust T to appear in the equation of virtual work.

Works are done by (1) the total weight $4W$ at G, (2) the weight W' at C and (3) thrust T of rod EF. $\therefore$ Equation of virtual work is, $4W.\delta(AG)+W'\delta(AC)+T\delta(EF)=0$

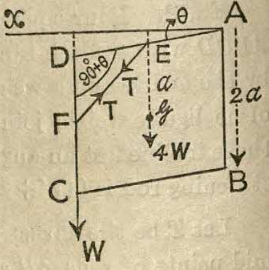
[We have $+T.\delta.(EF)$ and not $-T.\delta(EF)$ as T is a thrust and not a tension].

$$or,\quad 4W.\delta(2a\cos\sphericalangle)+W'\delta(4a\cos\sphericalangle)+T\delta(2a\sin\sphericalangle)=0$$

$$or,\quad 4W.(-2a\sin\sphericalangle)+W'(-4a\sin\sphericalangle)+T.2a\cos\sphericalangle=0.$$

$$or,\quad T=(4W+2W')\tan\sphericalangle.$$

Ex. 3. Four equal uniform rods each of weight W are freely jointed so as to form a square $ABCD$. The side AB is fixed in a vertical position with A uppermost. A weight W is suspended from the corner C, and the figure is kept in shape by an inextensible string joining the middle points of AD and DC. Find the tension of the string. [C. H. 1959]



We shall assume that in initial position of equilibrium AD makes an angle θ with the horizontal Ax ; actually θ is zero but to take the differentials we write it as a variable. Let $AB=2a$; and let T be the tension of the string EF, joining the middle points E and F of AD and DC. The total weight $4W$ of the 4 rods acts at G, their c.g. where EG is vertical and equal to a.

$\therefore$ Depth of G below $Ax=a+a\sin\theta$;

depth of $C=2a+2a\sin\theta$; $EF=2a\sin\left(\frac{90^\circ+\theta}{2}\right)$, since $\angle EDF=90^\circ+\theta$. Let the system be given a small virtual displacement in which θ changes to $(\theta+\delta\theta)$. Work is done by total weight $4W$ at G, weight W at C, and tension T.

Equation of virtual work is,

$$4W\delta(a+a\sin\theta)+W\delta(2a+2a\sin\theta)-T\delta.\left\{2a\cdot\sin\left(45^\circ+\frac{\theta}{2}\right)\right\}=0$$

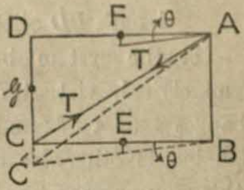
$$or,\quad 6Wa\cos\theta=T.a.\cos\left(45^\circ+\frac{\theta}{2}\right).$$

In position of equilibrium,

$$\theta=0,\quad\therefore\quad T=\frac{6W}{\cos 45^\circ}=6\sqrt{2}.W$$

Ex. 4. Four uniform rods are jointed to form a rectangle $ABCD$. AB is fixed in a vertical position with A uppermost, and the rectangle is kept in shape by a string joining AC. Show that the tension of the string is $\frac{W}{2}\cdot\frac{AC}{AB}$, where W is the total weight of the rods. [C. H. 1965]

Let E, F, G be the mid. points of BC, AD and DC and let ω, ω' be the weights of BC (or AD) and DC respectively. Then at E, F act weights ω, ω and at G acts weight ω'. Let T be the tension of the string AC. Let the system be given a small virtual displacement in which the rods AD, BC rotate about A and B respectively, each through a small angle θ ; then E and F each moves vertically down through $\frac{1}{2}AD.\theta$; D and C and $\therefore$ G, each moves vertically down through $AD.\theta$; and AC changes to AC' where $CC'=AD.\theta$. Since in $\triangle ADC$, when DC changes to DC', AC changes to AC' $\therefore$ differentiating the relation $AC^2=AD^2+DC^2$ in which AD is constant, we get



$$2AC\delta(AC)=2DC\delta(DC) \quad or, \quad \delta(AC)=\frac{DC}{AC}\cdot AD.\theta.$$

Equation of virtual work is,

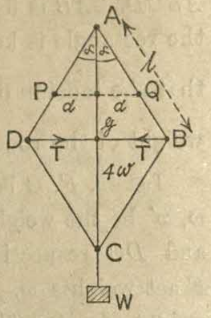
$$2.\omega(\tfrac{1}{2}AD.\theta)+\omega'.(AD.\theta)-T\cdot\frac{DC}{AC}\cdot AD.\theta=0$$

$$\therefore \quad T=(\omega+\omega')\cdot\frac{AC}{DC}=\frac{W}{2}\cdot\frac{AC}{AB}.$$

This sum can also be solved by the method followed in Ex. 3 above ; It will be a good exercise for the reader to solve Ex. 3 and Ex. 4 by both the methods shown.

Ex. 5. A rhombus $ABCD$, formed of four rods each of length l and weight ω, joined by smooth hinges, rests in equilibrium with AC vertical on two smooth pegs in a horizontal line distance $2d$ apart ; a weight W is attached to the lowest hinge C and B, D are joined by a string. If the angle at A is 2α, show that the tension of the string is

$$\tan\alpha\left[\frac{d}{2l}(W+4\omega)\operatorname{cosec}^3\alpha-(W+2\omega)\right] \qquad \text{[C. H. 1966]}$$



Let P, Q be the pegs where $PQ=2d$, and let T be the tension of string BD. The total weight $4w$ of the four rods acts at G, their c.g., where $AG=l\cos\alpha$.

$\therefore$ depth of G below fixed line PQ

$$=l\cos\alpha-d\cot\alpha.$$

depth of C below fixed line PQ

$$=2l\cos\alpha-d\cot\alpha$$

length BD of string $=2l\sin\alpha$

Let the system be given a small, symmetrical, virtual displacement which changes α to $\alpha+\delta\alpha$ and which changes the length of the string BD, but is otherwise consistent with the geometrical conditions (this may be done by pulling C vertically downwards through a small distance). Equation of virtual work is,

$$4w\delta(l\cos\alpha-d\cot\alpha)+W\delta(2l\cos\alpha-d\cot\alpha)-T\delta(2l\sin\alpha)=0$$

$$or,\quad 4w(-l\sin\alpha+d\operatorname{cosec}^2\alpha)+W(-2l\sin\alpha+d\operatorname{cosec}^2\alpha)$$
$$-T.2l\cos\alpha=0$$

$$\therefore\quad 2l\cos\alpha T=d(4w+W)\operatorname{cosec}^2\alpha-2l\sin\alpha(W+2w)$$

$$\therefore\quad T=\tan\alpha\left[\frac{d}{2l}(4w+W)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

If there be no string BD then $T=0$,

$$\therefore\quad \sin^3\alpha=\frac{d(4w+W)}{2l(W+2w)},\text{ gives position of equilibrium.}$$

[Nagpur 1936]

If $w=0$, then $T=W\left[\frac{d}{2l}\sec\alpha\operatorname{cosec}^2\alpha-\tan\alpha.\right]$ [C. H. 1957]

Ex. 6. *ABCD is a rhombus of freely jointed rods lying flat on a smooth table and P, Q are the middle points of AB, AD. Prove that if the system is held in equilibrium by light strings joining P to Q and A to C, the tensions in these strings are in the ratio of 2BD to AC.*

Let T, T' be the tensions of the strings PQ and AC. Giving the system a small displacement in which it remains on the table and the lengths of the strings are altered only, we get the equation of virtual work

$$-T\delta(PQ)-T'\delta(AC)=0\ldots\ldots(1)$$

Now $PQ=\frac{1}{2}BD$ and we know, from geometry,

$$AC^2+BD^2=4AB^2\equiv\text{const.}$$

$$\therefore\quad AC\delta(AC)+BD\delta(BD)=0$$

From (1) $\frac{1}{2}T\delta(BD)+T'\delta(AC)=0$

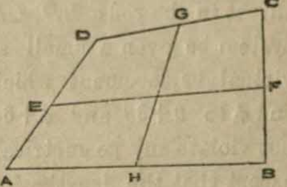
$$\therefore\quad \frac{T'}{AC}=\frac{\frac{T}{2}}{BD}\quad or\quad T:T'=2BD:AC.$$

(C) Rods forming a quadrilateral.

Ex. 7. The middle points of the opposite sides of a quadrilateral formed by four freely jointed weightless bars are connected by two light rods of lengths l and l' in a state of tension. If T, T' be the tensions in these rods, prove that $\frac{T}{l}+\frac{T'}{l'}=0$.

[C. H. 1943]

Let $ABCD$ be the quadrilateral and EF, GH the light rods joining mid. points of opposite sides; and let T, T' be the tensions in the rods EF and GH of lengths l, l' respectively. Imagine the system is given a small deformation in which no geometrical conditions are violated except that the lengths of the rods EF, GH undergo slight alterations (this can be done for example, by pushing A and C towards each other along AC). Equation of virtual work is



$$-T\delta(EF)-T'\delta.(GH)=0\quad or,\quad T\delta l+T'\delta l'=0\text{......}(1)$$

We have to find the geometrical relation between EF and GH. If O be the point of intersection of EF and GH (not shown in fig.), then from the Δ's AOB, COD, by applying the theorem of Appolonius, we have, $OA^2+OB^2=2OH^2+2AH^2$, $OC^2+OD^2=2OG^2+2CG^2$; adding, $OA^2+OB^2+OC^2+OD^2=GH^2+\frac{1}{2}AB^2+\frac{1}{2}CD^2$. Similarly considering the Δ's OBC, ODA we have $OA^2+OB^2+OC^2+OD^2=EF^2+\frac{1}{2}BC^2+\frac{1}{2}DA^2$.

$$\therefore\quad GH^2+\tfrac{1}{2}AB^2+\tfrac{1}{2}CD^2=EF^2+\tfrac{1}{2}BC^2+\tfrac{1}{2}DA^2$$

$$or,\quad l^2-l'^2=\tfrac{1}{2}(AB^2+CD^2-BC^2-AD^2)$$

Differentiating, $2l\delta l - 2l'\delta l' = 0$, [as lengths of the bars are unchanged.]

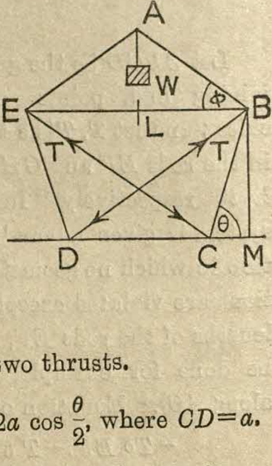
$$\text{or, } \quad l\delta l = l'\delta l'$$

$\therefore$ From (1), we have, $\frac{T}{l} + \frac{T'}{l'} = 0$

The result obtained shows that both T, T' cannot be tensions. If one is a tension, the other must be a thrust.

(D) Rods forming a pentagon.

Ex. 8. Five weightless rods form a regular pentagon $ABCDE$, and the framework is stiffened by other weightless rods BD, CE. The system is placed in a vertical plane with CD on a horizontal table, and a weight W is hung from A. Prove that the thrust in BD or CE is $W \cot \frac{\pi}{5}$. [C. H. 1951]



In position of equilibrium let BC make with DC produced an angle θ and let $\angle ABE = \phi$. Let T be the thrust in the rods BD, CE. Let the system be given a small symmetrical virtual displacement which changes θ and ϕ to $\theta + \delta\theta$ and $\phi + \delta\phi$. It does not violate any geometrical conditions except that the lengths of the rods BD, EC are slightly altered.

$\therefore$ work is done only by W and the two thrusts.

Now $\angle CBD = \angle CDB = \frac{\theta}{2}$ $\therefore$ $BD = 2a \cos \frac{\theta}{2}$, where $CD = a$.

Also height of A above $DC = AL + BM = a \sin \phi + a \sin \theta$

$\therefore$ Equation of virtual work is,

$$-W\delta(a \sin \phi + a \sin \theta) + 2.T\delta\left(2a \cos \frac{\theta}{2}\right) = 0$$

$$\text{or, } \left(W \cos \theta + 2T \sin \frac{\theta}{2}\right)\delta\theta + W \cos \phi\, \delta\phi = 0 \text{......(1)}$$

Now $EB = 2BL = 2a \cos \phi$ and $EB = DC + 2CM = a + 2a \cos \theta$

$$\therefore \quad 2a \cos \phi = a + 2a \cos \theta$$

$$\therefore \quad \sin \phi\, \delta\phi = \sin \theta\, \delta\theta \text{......(2)}$$

From (1) and (2), we have, $T=-\dfrac{W\sin(\theta+\phi)}{2\sin\frac{\theta}{2}\sin\phi}$.

In a regular pentagon, each interior angle $=\dfrac{2.5-4}{5}\cdot\dfrac{\pi}{2}=\dfrac{3\pi}{5}$

$\therefore\quad \theta=\pi-\dfrac{3\pi}{5}=\dfrac{2\pi}{5}$ and $\phi=\dfrac{3\pi}{5}-\dfrac{2\pi}{5}=\dfrac{\pi}{5}$

$$\therefore\quad T=-\frac{W.\sin\frac{3\pi}{5}}{2\sin\frac{\pi}{5}\cdot\sin\frac{\pi}{5}}=-\frac{W.\sin\frac{2\pi}{5}}{2\sin\frac{\pi}{5}\sin\frac{\pi}{5}}=-W\cot\frac{\pi}{5}.$$

The negative sign shows that T is not a thrust but is really a tension. Hence the tension in the rod BD or CE is $W\cot\frac{\pi}{5}$.

(E) Rods forming a hexagon.

Ex. 9. Six equal bars smoothly jointed form a regular hexagon $ABCDEF$ which is kept in shape by vertical strings joining the middle points of BC, CD and AF, FE, respectively, the side AB being held horizontal and upper-most. Prove that the tension of each string is three times the weight of a bar.

[Lucknow, 1956]

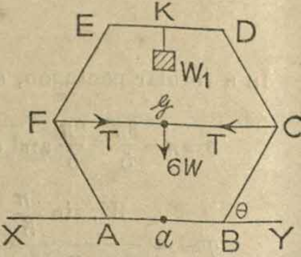
Let T be the tension of each of the strings PQ, RS where P, Q, R, S are the middle points. In position of equilibrium let BC make an angle θ with AB produced and let W be the weight of each bar. Their total weight $6W$ acts at their c. g. which is G, the middle point of CF. Depth of G below fixed line $AB=a\sin\theta$, where $AB=a$. PQ or $RS=\frac{1}{2}BD=\frac{1}{2}.\,2a\sin\theta=a\sin\theta$. Let the system be given a small, symmetrical, virtual displacement in which θ changes to $\theta+\delta\theta$. Work is done only by the total weight $6W$ at G and the two tensions T, T of the strings PQ and RS.

$\therefore$ Equation of virtual work is,

$6W\delta(a\sin\theta)-2.T\delta(a\sin\theta)=0\quad\therefore\quad T=3W.$

Ex. 10. A regular hexagon is composed of six equal heavy rods freely jointed together, and two opposite angles are connected by a string, which is horizontal, one rod being in contact with a horizontal plane; at the middle point of the opposite rod is placed a weight W_1; if W be the weight of each rod, show that the tension of the string is $\dfrac{3W+W_1}{\sqrt{3}}$. [C. H. 1946]

Let $ABCDEF$ be the regular hexagon of which the side AB is placed on a horizontal plane XY. In position of equilibrium let $\angle CBY=\theta$ and let $AB=a$ and T be the tension of the string CF. W_1 is placed at K, mid point of side DE. The total weight $6W$ of the rods acts at their c. g. G, the mid point of CF. Height of G above $AB=a \sin\theta$; height of K above $AB=2a \sin\theta$ and $CF=a+2a\cos\theta$. Let the system be given a small, symmetrical, virtual displacement in which θ changes to $\theta+\delta\theta$.



∴ Equation of virtual work is,

$$-6W.\delta(a\sin\theta)-W_1.\delta(2a\sin\theta)-T.\delta(a+2a\cos\theta)=0$$

or, $-(6W+2W_1)a\cos\theta\delta\theta+T.2a\sin\theta\delta\theta=0$

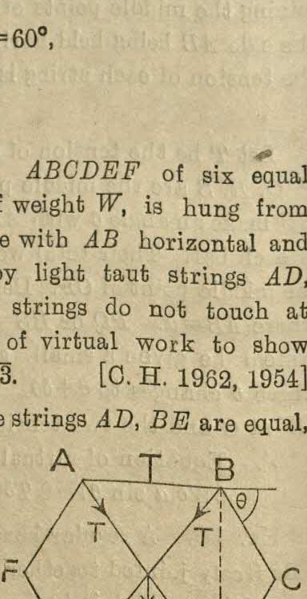
or, $T=(3W+W_1)\cot\theta.$

Since in position of equilibrium, $\theta=60°$,

we have $T=\dfrac{3W+W_1}{\sqrt{3}}.$

Ex. 11. A rectangular hexagon $ABCDEF$ of six equal uniform smoothly jointed rods, each of weight W, is hung from the mid point of AB in a vertical plane with AB horizontal and uppermost. Its shape is maintained by light taut strings AD, BE, and it may be assumed that the strings do not touch at their mid points. Apply the principle of virtual work to show that the tension in each string is $W\sqrt{3}$. [C. H. 1962, 1954]

From symmetry the tensions of the strings AD, BE are equal, say, each is T. The total weight $6W$ of the rods acts at their c. g. G whose depth below fixed side AB is $a\sin\theta$ if $a=AB$ and in position of equilibrium BC makes an angle θ with AB. Let the system be given a small virtual displacement by pulling DE vertically down so that the points D and E move on the verticals BD and AE



through the fixed points B and A, DE remaining parallel to AB. Equation of virtual work is $6W.\delta(a\sin\theta)-2.T.\delta(AD)=0$. To find $\delta\,(AD)$ note that in $\Delta\,ABD$ due to the displacement conceived D moves vertically down BD and consequently AD changes, AB remaining constant.

Since $AD^2=AB^2+BD^2=a^2+(2a\sin\theta)^2$,

$$\therefore\quad 2AD.\delta(AD)=8a^2\sin\theta\cos\theta\delta\theta.$$

$$\therefore\quad 3Wa\cos\theta=T.\frac{4a^2\sin\theta\cos\theta}{AD}.$$

$$or,\quad T=\frac{3W}{4a\sin\theta}\cdot AD.$$

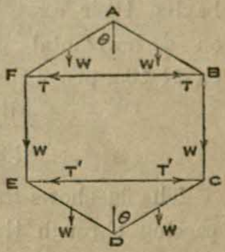
In position of equilibrium, $\theta=60^\circ$

$$\therefore\quad AD^2=a^2+4a^2\cdot\left(\frac{\sqrt{3}}{2}\right)^2=4a^2\quad or,\quad AD=2a$$

$$\therefore\quad T=\frac{3W.2}{4a.\sqrt{3}}\cdot 2a=W\sqrt{3}.$$

Ex. 12. Six equal heavy rods, freely hinged at the ends, form a regular hexagon $ABCDEF$, which when hung up by the point A, is kept from altering the shape by two light rods BF, CE. Prove that the thrusts of the rods are as 5 : 1, and find their magnitudes. [M. T. ; C. H.]

Let the length of each rod be a and W its weight and let θ be the inclination of either of the inclined rods to the vertical. Let T, T' be the thrusts in the rods BF and CE. Let the system be given a small symmetrical, virtual displacement in which θ changes to $\theta+\delta\theta$, D moves vertically down, and no geometrical conditions are violated except that the lengths of the rods BF and CE are slightly altered. The total weight of the rods, $6W$, acts at their c. g, the centre of the hexagon, whose depth below the fixed point A is $\left(a\cos\theta+\frac{a}{2}\right)$. And $CE=BF=2a\sin\theta$.



The equation of virtual work is

$$6W.\delta\left(a\cos\theta+\frac{a}{2}\right)+T.\delta(2a\sin\theta)+T'.\delta(2a\sin\theta)=0$$

$$or,\quad T+T'=3W\tan\theta\quad \ldots\quad \ldots\quad (1)$$

To find T', let the system be given a small virtual displacement in which the upper three rods forming the triangle ABF are kept fixed and the point D moves down vertically through an infinitesimal distance. The rods FE and BC turn through a small angle round their ends F and B respectively, and their mid points moving horizontally their weights do no work. The c. g. of the two lower rods ED, DC are slightly lowered and the work done by their weights is $2W\delta\left(\frac{a}{2}\cos\theta\right)$

$\therefore$ Equation of virtual work is

$$2W.\delta\left(\frac{a}{2}\cos\theta\right)+T'\delta(2a\sin\theta)=0$$

or, $$T'=\frac{W}{2}\tan\theta \quad \ldots \quad \ldots \quad (2)$$

$\therefore$ From (1) and (2), $T=\frac{5W}{2}\tan\theta$. Hence $T:T'=5:1$

In position of equilibrium, $\theta=60°$

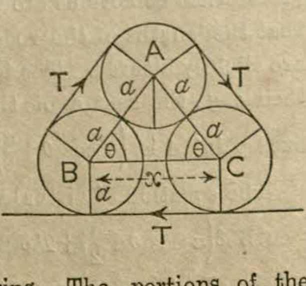
$$\therefore \quad T'=\frac{\sqrt{3}}{2}\cdot W. \text{ and } T=\frac{5\sqrt{3}}{2}\cdot W.$$

Miscellaneous Problems

Ex. 1. An endless string of length $2\pi a+l$ passes round three equal smooth cylinders, each of weight W and radius a, having their axes horizontal and parallel, and two of these rest on a horizontal plane, the third lying between them. If $6a<l<8a$, prove that the tension of the string is

$$\frac{\frac{1}{2}W(l-4a)}{\{l(8a-l)\}^{\frac{1}{2}}}. \qquad \text{[C. H. 1955]}$$

The fig. shows the section of the cylinders by the vertical plane passing through their c.g.'s A, B, C, and perpendicular to their axes. The cylinders B, C lie on the plane and A lies on them. We have $AB=AC=2a$; let $BC=x$ and let $\angle ABC=\angle ACB=\theta$. Then $x=4a\cos\theta\ldots(1)$ and height of A above plane$=2a\sin\theta+a$.



We have to find the length of the string. The portions of the string touching the cylinders B, C and A have the lengths $(\pi-\theta)a$, $(\pi-\theta)a$, and $2\theta.a$ whose sum is $2\pi a$. The lengths of the straight portions are x, $2a$, $2a$ whose sum is $x+4a$. Therefore the length of the string$=2\pi a+x+4a=2\pi a+l$, in position of equilibrium ; then $x=l-4a$......(2).

$\therefore$ $x>2a$ but $<4a$. This shows that B, C do not touch and A lies on B and C. Let the system be given a small symmetrical virtual displacement in which A moves down vertically and B, C move sideways equally. No geometrical constraints are violated except that the length of the string is altered. Let T be the tension of the string. Since B, C move horizontally their vertical weights do no work. Work is done only by the weight of A and tension T. Therefore the equation of virtual work is

$$-W\delta(2a\sin\theta+a)-T\delta(2\pi a+4a\cos\theta+4a)=0 \text{ [using (1)].}$$

$$\therefore\quad T=\tfrac{1}{2}W\cot\theta.$$

Since in position of equibrium,

$$\cos\theta=\frac{x}{4a}=\frac{l-4a}{4a},\ \sin\theta=\frac{\sqrt{l(8a-l)}}{4a}$$

$$\therefore\quad T=\tfrac{1}{2}W.\frac{(l-4a)}{\sqrt{l(8a-l)}}.$$

Ex. 2. A heavy uniform rod AB, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclinations α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that $\tan\theta=\frac{1}{2}(\cot\alpha\sim\cot\beta)$.

In position of equilibrium, the rod AB must lie in the vertical plane which cuts both planes in lines of greatest slope. Let GL be the height of G, the c g. of the rod, above O where the planes intersect. Then $GL=\frac{1}{2}(OA\sin\alpha+OB\sin\beta)$. But from $\triangle OAB$, we have,

$$\frac{OA}{\sin(\beta-\theta)}=\frac{OB}{\sin(\alpha+\theta)}=\frac{2a}{\sin(\alpha+\beta)}.$$

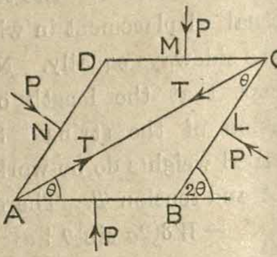
$$\therefore\quad GL=a\,[\sin\alpha\sin(\beta-\theta)+\sin\beta.\sin(\alpha+\theta)]\div\sin(\alpha+\beta).$$

Give the rod a small displacement in the vertical plane in which it lies without violating any constraint, and which changes θ to $\theta+\delta\theta$. Equation of virtual work is $W.\delta(GL)=0$

or $-\sin\alpha\cos(\beta-\theta)+\sin\beta\cos(\alpha+\theta)=0$

Simplifying result follows.

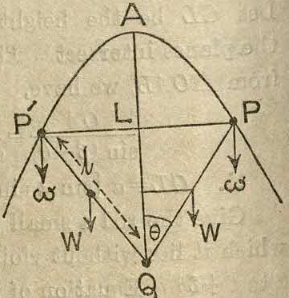
Ex. 3. A frame consists of five bars forming the sides of a rhombus $ABCD$ with the diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in AC is $\frac{P\cos 2\theta}{\sin\theta}$ where θ is the angle BAC. The bars are weightless.



Let L, M, N be the mid-points of BC, CD and DA. Keeping AB fixed, give the system a slight deformation in which the lengths of the rods are not violated except that of AC and rods BC, AD turn round their ends B and A respectively through a small angle say $\sphericalangle$. L moves $\perp$ to BC through a small distance $\frac{1}{2}a\sphericalangle$ where $a=AB$. $\therefore$ Work done by P at $L=\frac{1}{2}a\sphericalangle.P$. Similarly work done by P at $N=-\frac{1}{2}a\sphericalangle.P$. To find the work done by P at M, we have to find the vertical displacement of M. Let z=height of M above AB and T=tension in AC. Then $z=BC\sin 2\theta=a\sin 2\theta$ and $AC=2a\cos\theta$. $\therefore$ Total work done$=\frac{1}{2}a\sphericalangle.P-\frac{1}{2}a\sphericalangle.P-P\delta z-T\delta(AC)$ $=-P\delta(a\sin 2\theta)-T\delta(2a\cos\theta)$.

$$\text{Hence } -P.a.\,2\cos 2\theta+T.\,2a\sin\theta=0 \quad \therefore\quad T=P.\frac{\cos 2\theta}{\sin\theta}.$$

Ex. 4. Two uniform straight rods PQ, $P'Q$, in all respects alike, are smoothly jointed at Q and P, P' carry small rings which slide on a smooth fixed parabolic wire whose axis is vertical and vertex upwards. Prove that in the symmetrical position of equilibrium the angle either rod makes with the horizontal is $\sin^{-1}\left\{\frac{aW}{l(W+\omega)}\right\}$, where W is the weight of either rod, ω of either ring, l the length of either rod and $4a$ the latus rectum of the parabola.



Let $\angle PQA=\theta$; draw $PL \perp$ on axis AQ $\therefore$ $PL=l\sin\theta$, $QL=l\cos\theta$. Since $PL^2=4aAL$ $\therefore$ $AL=\frac{1}{4a}l^2\sin^2\theta$.

$\therefore$ Depth of P *or* P' below $A=\frac{1}{4a}l^2\sin^2\theta$; and depth of c.g.

of either rod$=AL+\frac{1}{2}LQ=\frac{1}{4a}\,l^2\sin^2\theta+\frac{1}{2}\,l\cos\theta$. Let the system be given a small symmetrical virtual displacement which changes θ to $\theta+\delta\theta$ but does not violate any constraint. Equation of virtual work is

$$2W.\delta\left(\frac{1}{4a}\,l^2\sin^2\theta+\tfrac{1}{2}l\cos\theta\right)+2\omega.\delta\left(\frac{1}{4a}\,l^2\sin^2\theta\right)=0$$

or, $(W+\omega)\frac{l^2}{a}.\sin\theta\cos\theta-lW\sin\theta=0$

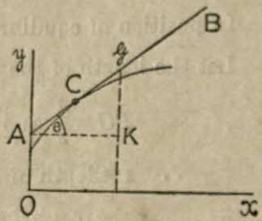
$\therefore$ Either $\theta=0$ *or*, $\theta=\cos^{-1}\left[\frac{aW}{l(W+\omega)}\right]$

$\therefore$ Each rod makes with the horizontal an angle equal to

$$\sin^{-1}\left[\frac{aW}{l(W+\omega)}\right].$$

Ex. 5. A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; If the beam is in equilibrium in any position, find the equation to the curve.

Let AB be the rod resting against the vertical wall at A and touching the curve at C. Take as y-axis the section of the wall by the vertical plane containing the curve and take any point O on it as origin. Let z be the height of G, the middle point of the rod above Ox. Let the rod be given a small virtual displacement in the xy-plane consistent with the geometrical conditions. Equation of virtual work is $W\delta z=0$



$\therefore$ $z=\text{const}=K$, say. Thus in all positions of equilibrium G is at a constant height and locus of G is a horizontal line. To find the equation of the curve we note that it is the envelope of AB, which in all its positions is a tangent to the curve. Let AB make an angle θ with Ox, and let $AB=2l$. Then G is $(l\cos\theta, K)$

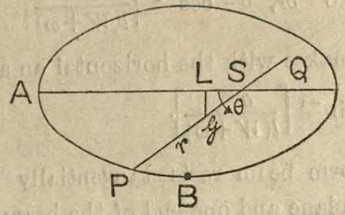
$\therefore$ Equation of AB is $y-K=\tan\theta(x-l\cos\theta)$

or, $y-K=x\tan\theta-l\sin\theta$. Differentiating *w.r.* to θ, we have, $0=x\sec^2\theta-l\cos\theta$ $\therefore$ $x=l\cos^3\theta$ and

$\therefore$ $y-K=-l\sin^3\theta$. $\therefore$ $x^{\frac{2}{3}}+(y-K)^{\frac{2}{3}}=l^{\frac{2}{3}}$. This is the equation of the curve. It is an Astroid.

Ex. 6. A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal, and rests with its lower end on the quadrant of the curve which is furthest removed from the focus. Show that its length must be at least $\frac{3a}{4}+\frac{a}{4}\sqrt{1+8e^2}$, where a is the semi-major axis and e the eccentricity. [M. T.; C. H. 1941]

Let PQ be the rod, resting on the ellipse at P and passing through the ring at the focus S.



Polar equation of the ellipse referred to S as pole and SA as initial line is $\frac{l}{r}=1-e\cos\theta$......(1) where $l=a(1-e^2)$......(2)

In position of equilibrium let P be (r, θ) $\therefore$ $\angle ASP=\theta$, $SP=r$.

Let the length of the rod $PQ=d$ and let G be its mid. point.

$$\therefore \quad PG=\frac{d}{2}, \text{ and } SG=r-\tfrac{1}{2}d$$

$$\therefore \quad z=\text{depth of } G \text{ below } AS=GL$$

$$=(r-\tfrac{1}{2}d)\sin\theta.$$

Equation of virtual work for the rod is $W.\delta z=0$, where $W=$ wt. of the rod.

$$\therefore \quad \delta.[(r-\tfrac{1}{2}d)\sin\theta]=0$$

$$or, \quad (r-\tfrac{1}{2}d)\cos\theta\,\delta\theta+\sin\theta.\delta r=0$$

From (1), $-\frac{l}{r^2}\delta r=e\sin\theta\,\delta\theta$

$$\therefore \quad (r-\tfrac{1}{2}d)\cos\theta-\frac{e\sin^2\theta\, r^2}{l}=0$$

$$or, \text{ using (1)}: \quad (r-\tfrac{1}{2}d)\left(\frac{r-l}{r}\right)-\frac{r^2}{l}\left[e^2-\left(\frac{r-l}{r}\right)^2\right]=0$$

Simplifying, we have,

$$d=\frac{2r^2(r-l-e^2r)}{l(r-l)}=\frac{2r^2(r-a)(1-e^2)}{a(1-e^2)(r-l)} \qquad [\text{ from (2) }]$$

$$=\frac{2r^2}{a}\left(\frac{r-a}{r-l}\right) \text{ ......(3)}$$

Since P lies in quadrant AB, $SB<r<SA$ or, $a<r<a+ae$.

For each value of r between a and $a+ae$, the equation (3) gives a positive value of d and if this value of d be $\geqslant$ the value of r, then a rod of length d will be in equilibrium when lying along the radius vector of length r. Values of r for which $d \geqslant r$ satisfy the inequality $\frac{2r^2}{a}\left(\frac{r-a}{r-l}\right) \geqslant r$

or, $2r^2-3ar+al \geqslant 0$ $[\because \; r \geqslant a > l]$

or, $2(r-\alpha)(r-\beta) \geqslant 0$, where $\alpha=\frac{3a}{4}-\frac{a}{4}\sqrt{1+8e^2}$,

$$\beta=\frac{3a}{4}+\frac{a}{4}\sqrt{1+8e^2}.$$

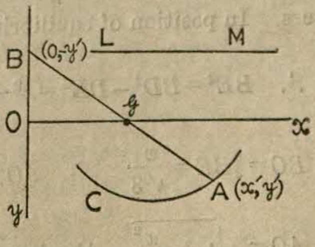
Since $r \geqslant a > \alpha$ $\therefore$ $d \geqslant r$ only if $r \geqslant \beta$.

When $r=\beta$, $d=r=\beta$ and when $r>\beta$, $d>r>\beta$.

$\therefore$ Least value of d is β.

$\therefore$ Least length of rod reqd. $=\frac{3a}{4}+\frac{a}{4}\sqrt{1+8e^2}$.

Ex. 7. One end of a beam rests against a smooth vertical wall and the other end on a smooth curve in a vertical plane perpendicular to the wall; if the beam rests in all positions, show that the curve is an ellipse whose major axis lies along the horizontal line described by the centre of gravity of the beam.



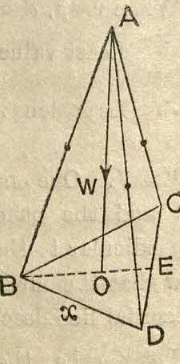
Let Oy be the section of the vertical wall by the vertical plane containing the curve CA. Let G be the c. g. of the rod AB and W its wt. Suppose in a position of equilibrium the depth of G below some fixed horizontal line LM is z. Then for a small virtual displacement which violates no geometrical conditions, the equation of virtual work is $W\delta z=0$ or $z=$const. $\therefore$ In all different positions of equilibrium of rod AB, its c. g. G is at a constant distance from LM $\therefore$ Locus of G is a horizontal line. Take this line as the axis of x and Oy as the axis of y. Let co-ordinates of A be (x', y'). Since middle point G of AB is on Ox $\therefore$ B is $(0, -y')$.

$$\therefore\ (x'-0)^2+(y'+y')^2=AB^2=l^2, \text{ if length of rod } AB=l$$

$$or\ \ x'^2+4y'^2=l^2.$$

$\therefore$ Locus of (x', y'), which is the given curve, is $x^2+4y^2=l^2$, an ellipse, whose major axis is Ox, the locus of G.

Ex. 8. A tripod of three uniform light rods AB, AC and AD, each of length $2a$ stands with the feet B, C and D on level smooth ground at the vertices of an equilateral triangle of side a. A weight W is suspended from A and the tripod is prevented from slipping by three light strings joining the middle points of the rods. Apply the principle of virtual work to show that tensions in these strings are $\dfrac{2W}{3\sqrt{33}}$. [C. H. 1965]



Let the fig. show the position of equilibrium of the system. Let AO be the perpendicular from A on plane BCD. Then O is the circumcentre of $\triangle BCD$ and as this triangle is equilateral O is the centre of it. We give the system a small symmetrical displacement in which A moves vertically down along AO, B, C and D move through equal distances along OB, OC and OD so that $\triangle BCD$ remains equilateral, but the length of its sides alters. In position of equilibrium, let $BD=x$

$$\therefore\ BE^2=BD^2-DE^2=x^2-\frac{x^2}{4}=\tfrac{3}{4}x^2\ \ or,\ BE=\frac{\sqrt{3}}{2}x$$

$$BO=\tfrac{2}{3}BE=\frac{x}{\sqrt{3}}.\quad \therefore\ AO^2=AB^2-BO^2=4a^2-\frac{x^2}{3}.$$

$AO=\sqrt{4a^2-\dfrac{x^2}{3}}.$ Also length of a string $=\frac{1}{2}x$.

If T=tension in each string, equation of virtual work is

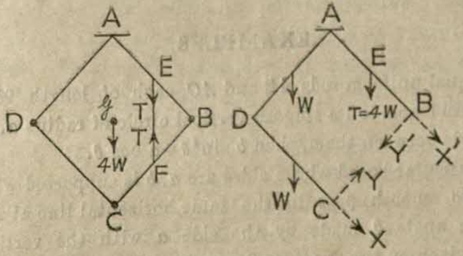
$$-W\delta(AO)-3.T\delta\left(\frac{x}{2}\right)=0$$

$$or,\ \ T=\frac{2W\sqrt{3}.x}{9\sqrt{12a^2-x^2}}.$$

In position of equilibrium $x=a$ $\therefore$ $T=\dfrac{2W}{3\sqrt{33}}$.

Ex. 9. A square frame $ABCD$ of four equal jointed rods hangs from A, the shape being maintained by a string joining mid-points of AB, BC. Prove that the ratio of the tension of the string to the reaction at C is $\dfrac{8}{\sqrt{5}}$. [C. H. 1948]

Let W=wt. of each rod and a its length. To find tension T of the string EF :—



Let $\angle BAC=\theta$, and let G be the c. g. of the 4 rods. Give the system a small virtual displacement in which θ changes to $\theta+\delta\theta$ and no constraints are violated except that the length of the string alters. Equation of virtual work is $4W\delta(AG)-T\delta(EF)=0$

or, $4W\delta(a\cos\theta)-T\delta(a\cos\theta)=0$ or, $T=4W$.

To find the reaction at C :—Remove the rod BC and apply at the ends B of AB and C of DC the actions of the rod BC at these ends and at E the tension $4W$. Let the action at C be represented by a force X along DC and a force $Y \perp$ to DC and that at B by X', Y' along AB and $\perp$ to AB. Then the frame is in equilibrium as before. Consider the rod DC and give it an anti-clockwise rotational displacement about D through a small angle θ. Equation of virtual work for rod DC gives $Ya\theta-W.\cos 45^\circ.\frac{a}{2}\theta=0$ or, $Y=\dfrac{W}{2\sqrt{2}}$. To find X, give the system formed by the rods AD, DC a small displacement in which A is kept fixed and AD rotates anti-clockwise round A through a small angle θ, and C moves in the direction DC *i.e.* perpendicularly to AD. Work done by wt. of rod $AD = W \cos 45^\circ \times \frac{a}{2}\theta$; work done by wt. of rod $DC = W \cos 45^\circ \times a\theta$ [$\because$ D and $\therefore$ rod DC moves in the direction DC through a distance $a\theta$] and work done by force $X = X.a\theta$; Y does no work. $\therefore$ Equation of virtual work is,

$$W.\frac{1}{\sqrt{2}}.\frac{a\theta}{2} + W.\frac{1}{\sqrt{2}}.a\theta + Xa\theta = 0 \quad \therefore \quad X = -\frac{3}{2\sqrt{2}}W$$

$$\therefore \quad R = \text{reaction at } C = \sqrt{X^2+Y^2} = \frac{W}{2\sqrt{2}}.\sqrt{10} = \frac{W}{2}\sqrt{5}.$$

$$\therefore \quad \frac{T}{R} = \frac{8}{\sqrt{5}}.$$

EXAMPLES

1. Two equal uniform rods AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that, if 2θ be the angle between them, then $b \sin^3\theta = a \cos\theta$.

2. A rectangular board whose sides are a, b is supported with its plane vertical on two smooth pegs in the same horizontal line at a distance c ; prove that the angle θ made by the side a with the vertical when in equilibrium is given by the equation $2c \cos 2\theta = b \cos\theta - a \sin\theta$. [W]

3. A smooth circular cylinder of radius b is fixed parallel to a smooth vertical wall with its axis at a distance c from the wall. A smooth uniform heavy rod of length $2a$ rests on the cylinder with one end on the wall and in a plane perpendicular to the wall, show that its inclination θ to the horizontal is given by $a \cos^3\theta + b \sin\theta = c$.

4. A rod AB is movable about a joint at A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is $\frac{w \cos\alpha \cos\beta}{2 \sin(\alpha+\beta)}$, where w is the weight of the rod and α, β the inclinations of the rod and the string to the horizontal. [Allahabad 1951]

5. Two equal light rods AOB, COD, freely jointed at O, their middle point, are at rest in a vertical plane with their ends B, C on a smooth horizontal table. A string to the ends of which equal weights are attached passes over A and D. Show that in the position of equilibrium the angle between the rods is $\tan^{-1}\frac{4}{3}$.

6. Two heavy rings slide on a fixed smooth parabolic wire whose axis is horizontal and the rings are connected by a string which passes over a smooth peg at the focus. Prove that in the position of equilibrium the depths of the rings below the axis of the parabola are proportional to their weights. [Allahabad]

7. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$.

8. A lamina in the form of an isosceles triangle rests with its plane vertical and its two equal sides each in contact with a smooth peg, the pegs being in the same horizontal plane; prove that the axis of the triangle makes with the vertical the angle zero or $\cos^{-1}\left(\frac{h\sin\alpha}{3c}\right)$, h being the length of the axis, α the vertical angle, and c the distance between the pegs. [W]

9. Two uniform rods AB, BC of weights W and W' are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is $\frac{(W+W')\cos A\cos C}{\sin B}$. [Lucknow, 1952]

10. Two uniform heavy rods AB, CD, such that $AB = 2CD$, are connected by a smooth hinge, which joins C to the middle point of AB. The system rests in a vertical plane with B, D on a smooth horizontal table and equilibrium is maintained with CD perpendicular to AB by means of a tight string joining the mid points of BC, CD. If the weights of AB, CD are $2w$, w respectively, find the tension of the string. [C. H. 1952]

11. A step ladder has pair of equal legs which are joined by a hinge at the top, and are connected by a cord attached at one-third of the distance from lower end to the top. If the weight of each leg of the ladder be W_1 and acts at their middle points and if a man of weight W is two-thirds the way up the ladder, show, by the principle of virtual work, that the tension in the cord is $\frac{1}{2}(W + \frac{3}{2}W_1)\tan\alpha$, α being the inclination of each leg to the vertical. [Nagpur]

12. Four equal heavy uniform rods, each of weight W, are freely jointed to form a rhombus $ABCD$ and is suspended by the joint A. A light rod connects the middle points of AB, AD so that the rhombus cannot collapse. Prove that the thrust of the light rod is $4W\tan\alpha$, where the angle BAC is α.

13. A square framework formed of heavy uniform rods of equal weight W jointed together is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod. [C. H. 1947, Allahabad '53]

14. Four rods are jointed together to form a parallelogram and the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths. [Allahabad 1955]

If T_1, T_2 be the tensions, a, b the adjacent sides of the paralleogram and α the angle between them, show that $\cos\alpha=\frac{a^2+b^2}{2ab}\cdot\frac{T_1^2\sim T_2^2}{T_1^2+T_2^2}$.

15. A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the rod is $\frac{2W(2b^2-a^2)}{b\sqrt{4b^2-a^2}}$.

16. $ABCD$ is a rhombus formed of light rods loosely jointed together; OB, OD are two equal rods jointed at O, B and D. If O is connected to A and C by strings in tension prove that the tension of each string is inversely proportional to its length.

17. A rhombus $ABCD$ is formed of four equal uniform rods freely jointed together and suspended from the point A; it is kept in position by a light rod joining the mid points of BC and CD; if T be the thrust in this rod and W the weight of the rhombus, prove that $T=W\tan\frac{A}{2}$.

18. A rhombus $ABCD$ of loosely jointed rods is in a horizontal plane with the rod BC fixed in position. The middle points of AD, DC are joined by a string which is kept taut by a couple L applied to rod AB. Find the tension of the string.

19. Four uniform rods are freely jointed at their extremities to form a parallelogram $ABCD$, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the whole weight.

20. A parallelogram $ABCD$ formed of four uniform rods freely jointed at the corners rests in equilibrium in a vertical plane with AB fixed horizontally. A is attached to the opposite corner C by a light string of length l. AC is the shorter diagonal and α is the acute angle of the parallelogram. Show that the tension of the string is $\frac{Wl\cot\alpha}{2a}$, where W= total weight of the four rods and $AB=a$. [C. H. 1958]

21. A frame $ABCD$ is formed of four light rods, each of length a, freely jointed together, it rests with AC vertical and the rods BC, CD in contact with fixed frictionless supports E, F in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that, when a weight W is placed on the highest joint A, it produces in BD a thrust of magnitude R where $Rb^2(4a^2-b^2)^{\frac{1}{2}}=W(2a^2c-b^3)$. [M. T.]

22. A rhombus $ABCD$ is formed of four light rods, smoothly jointed at their ends. A is connected with C and B with the middle point of CD, by tight strings. Show that if these strings are of equal length, the tension of one of them is double that of the other.

23. A square $ABCD$ formed of light rods, loosely jointed, has the side AB fixed. The middle points of AB, BC are joined by a string which is kept taut by a force P acting at the middle point of AD parallel to AB. Show that the tension of the string is equal to $P\sqrt{2}$.

24. A rhombus $ABCD$ formed of four uniform freely jointed rods, each of weight W and length a rests symmetrically in a vertical plane with AB, AD in contact with two smooth pegs in the same horizontal plane at a distance $2c$ apart, the vertex A being downwards. It is kept from collapsing by a light string BD. Prove that the tension of the string is

$$\frac{2W\left(a\sin\frac{A}{2}-c\operatorname{cosec}^2\frac{A}{2}\right)}{a\cos\frac{A}{2}}.$$

25. A quadrilateral $ABCD$ formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is $(\omega+\omega')\sin^2\theta+\omega'$, where ω is the weight of an upper rod and ω' of a lower rod and 2θ is equal to the angle BAD. [C. H. 1963]

26. An endless chain of weight ω rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be 2α. [Delhi 1950]

27. A smooth cone of weight W stands inverted in a circular hole with its axis vertical. A string is wrapped twice round the cone just above the the hole and pulled tight. What must be the tension in the string so that it will just raise the cone ?

28. A smooth paraboloid of revolution is fixed with its axis vertical and vertex upwards ; on it is placed a heavy elastic string of unstretched length $2\pi c$; when the string is in equilibrium show that it rests in the form of a circle of radius $\dfrac{4\pi ac\lambda}{4\pi a\lambda-cW}$, where W is the weight of the string, λ its modulus of elasticity and $4a$ the latus rectum of the generating parabola.

29. A regular pentagon $ABCDE$ is formed of five uniform heavy rods each of weight W freely jointed at their extremities. It is freely suspended from A and is maintained in its regular pentagonal form by a light rod joining B and E. Prove that the stress in this rod is $W\cot 18^\circ$. [London]

30. Five equal uniform rods, each of weight W, are freely jointed together to form a pentagon $ABCDE$, which is suspended from the joint A and maintained in the shape of a regular pentagon by two strings joining A to C and D. Show that the tension of either string is $2W\cos 18^\circ$.

31. A regular hexagon $ABCDEF$ consists of six equal rods, which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table ; if C and F be

connected by a light string, prove that its tension is $W\sqrt{3}$. Show that the action at the joint E is $\frac{W}{2}\sqrt{\frac{7}{3}}$ (W = weight of a rod). [Allahabad]

32. Six equal rods, each of weight W, are freely jointed at their ends to form a regular hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Prove that its tension is $3W$.

33. Six equal heavy beams are freely jointed at their ends to form a hexagon and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Show that its tension is $6W \cot \theta$, where W is the weight of each beam.

34. Four equal uniform rods of weight W are freely jointed so as to form a square $ABCD$ which is suspended from A and is prevented from collapsing by an inextensible string joining the middle points of AB and BC. Find the reaction at B.

35. Four rods, equal and uniform, rest in a vertical plane in the form of a square with a diagonal vertical and the two upper rods resting on two smooth pegs in a horizontal line. Show that the pegs must be at the middle points of the rods and find the actions at the hinges. [London]

MISCELLANEOUS EXAMPLES

1. A rectangular lamina $ABCD$ rests with sides AB, AD on two smooth pegs in a horizontal line; prove that if the distance between the pegs is half a diagonal of the rectangle, AB, AD bisect the angles between AC and the horizon.

2. Two smooth rods AB, AC are fixed in a vertical plane at right angles to one another and a heavy uniform rod BC can slide with its ends B, C on the rods AB, AC respectively. Show that the rod BC can rest in equilibrium at an inclination to the vertical double that of either of the fixed rods. [M. T.]

3. A triangular lamina ABC rests with its plane vertical, and with the sides AB, AC supported by smooth pegs D, E in a horizontal line. Prove that if $AD = p$, $AE = q$, then $b(q - p \cos A) - c(p - q \cos A) + 3(p^2 - q^2) = 0$.

4. Four equal jointed rods, each of length a, are hung from an angular point, which is connected by an elastic string with the opposite point. If the rod hangs in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is $\frac{a\sqrt{2}}{3}$.

5. A parallelogram $ABCD$ is formed of four rods jointed together. Equal and opposite forces P are applied inwards at A and C, and a string connects B and D. Prove that the tension of the string is $P \cdot \frac{BD}{AC}$.

6. A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.

7. If two equal uniform rods connected by a smooth hinge are placed over the circumference of a smooth vertical circle, apply the principle of virtual work to show that in the position of equilibrium the inclination of either rod to the vertical is given by $\operatorname{cosec}^2\theta \cot\theta = \frac{l}{a}$, where $2l$ and $2a$ are the length of a rod and the diameter of the circle respectively.

8. Two light rods AOC, BOD are smoothly hinged at O, a point at a distance c from each of the ends A, B which are connected by a string of length $2c \sin\alpha$. The rods rest in a vertical plane with the ends A and B on a smooth horizontal table. A smooth circular disc of radius a and weight ω is placed on the rods above O with its plane vertical so that the rods are tangents to the disc. Prove that the tension of the string is

$$\tfrac{1}{2}\omega\left[\frac{a}{c}\ \operatorname{cosec}^2\alpha + \tan\alpha\right].$$

9. Two equal rods AB, BC each of length l have a smooth joint at B and the ends A, C carry smooth light rings which can slide on the upper arc of an elliptic wire of axes $2a$, $2b$; the major axis $(2a)$ is vertical and the plane of the wire is also vertical. Show that if $b^2 < 2al < 2ab$, and the system is in equilibrium, each rod is either vertical or inclined to the vertical at an angle θ where $b^4 = l^2(4a^2\cos^2\theta + b^2\sin^2\theta)$.

10. A regular hexagon $ABCDEF$ composed of six equal and uniform rods each of weight W, freely jointed at their ends, is just balanced in a vertical plane. The rod AB is fixed horizontally, and two weightless rods connect A with the middle point of CD and B with the middle point of EF. Prove that the thrust of each weightless rod is $\frac{1}{2}W\sqrt{39}$.

11. Five equal uniform rods, each of weight W, are freely jointed to form a pentagon. It is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles θ and ϕ respectively with the vertical, prove that the tension of the string is to W as $(4\tan\theta + 2\tan\phi)$: $(\tan\theta + \tan\phi)$.

12. $ABCDEF$ is a regular hexagon formed of light rods smoothly jointed at their ends with a diagonal rod AD. Four equal forces P act inwards at the midpoints of the rods AB, CD, DE, FA and at right angles to them. Find the stress in the diagonal AD and state whether it is a tension or a thrust.

13. A heavy rod AB, of length $2l$, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions show that the polar equation of the curve, with C as origin, is $r = l + a \operatorname{cosec}\theta$.

[C. H. 1944]

14. A small heavy ring P slides on a smooth wire whose plane is vertical, and is connected by a string passed over a small pulley O in the plane of the curve with another weight W which hangs freely. If the ring is in equilibrium in all positions on the wire, show that the form of the latter must be a conic section whose focus is at the pulley.

15. Three equal light rods, each of length l, are smoothly jointed at one end to form a tripoid, and stand with their other ends on a horizontal smooth table, their feet being connected by three equal elastic strings. A load W is hung from the joint, and W is the modulus of elasticity of each string. If in the equilibrium position each rod makes an angle of 30° with the vertical, prove that the unstretched length of a string is $(\cdot 78)l$.

[C. H. 1948]

16. $ABCD$ is a square formed of four light rods jointed together, the diagonal AC being a fifth light rod. Weights P and Q are attached to the corners B, D respectively and the system is hung up by the corner A. Find the inclination of rod AC to the vertical and also the stress in it.

17. A rhombus $PQRS$ is formed by joining freely four equal uniform rods each of weight W. A fifth uniform rod LM of half the length and weight of one of the former rods is freely jointed at its ends to the middle points of QR and RS. If the system is suspended from the hinge P, show that the stress at each of the hinges at L, M is $\frac{1}{2}W\sqrt{30\frac{1}{3}}$.

STABLE AND UNSTABLE EQUILIBRIUM

4·1. Suppose a body or a system of bodies is in equilibrium under the action of any forces. If the body or the system be now displaced from its position of equilibrium into some neighbouring position and then set free, it may, under the action of the forces in the new position, remain in equilibrium in its new position or it may begin to move back to its original position of equilibrium or it may begin to move further away from its original position of equilibrium. In fhe first case the equilibrium is called *neutral*, in the second case it is called *stable* and in the last case it is called *unstable*. In cases where several displacements are possible it may happen that the equilibrium is stable for some displacements and unstable for others. In such a case we shall say that the equilibrium is on the whole unstable.

A rod PQ which can move freely about a smooth hinge at the end P will be in equilibrium when placed in the vertical position. When PQ hangs vertically with Q *below* P the equilibrium is stable for if PQ be displaced slightly and then set free it will begin to move towards its position of equilibrium but when placed in the vertical position with Q *above* P, the equilibrium is unstable for if it be slightly displaced and then set free, it will fall away from its position of equilibrium. A heavy uniform sphere at rest on a smooth horizontal plane is an example of a body in neutral equilibrium.

4·2. Co-ordinates of a body and of a system of bodies.

Since we are concerned with the positions of equilibrium of a body or of a system and their displacements to neighbouring positions it is necessary to find out how the position of a body or a system can be determined. The position of a particle in space is determined by its three cartesian co-ordinates x, y, z, referred to some axes. x, y, z are called the *co-ordinates of the particle*. Any set of quantities which will determine the position of the particle will be called its co-ordinates. The position of a rigid body is determined by the positions of any three points fixed in the body say A, B, C. This requires nine quantities, the nine cartesian co-ordinates of A, B, C. But since the three distances

AB, BC, CA are fixed, the nine quantities are not independent but are connected by three equations. Solving these equations we can express three of them in terms of the other six. These six quantities are independent and they determine the position of the body; they are called the *co-ordinates of the body.* Any arbitrary values given to these six co-ordinates will determine a position of the body and so the body is said to have six *degrees of freedom.* The position of the body can also be determined in other ways *e.g.* by the positions of any two points fixed in the body, say A and B, and the angle any plane fixed in the body makes with a plane fixed in space, say the xy-plane. We again find that six independent quantities are required for determining the position of the body. And it is always found that six quantities are necessary to fix the position of a rigid body. Thus a rigid body has six co-ordinates and six degrees of freedom.

The position of a system of rigid bodies, connected together and subject to certain constraints, is determined by the positions of the separate bodies which form the system. Let n be the total number of quantities required for determining the positions of the separate bodies. But these n quantities are not independent. Every connection and constraint gives rise to an equation involving these n quantities. Let m be the number of such equations. Solving these m equations we can express m of the quantities in terms of the other $(n-m)$ quantities. These $(n-m)$ quantities are independent and determine the position of the system. They are called its co-ordinates. Since each of these $(n-m)$ quantities can assume any arbitrary value, the system is said to have $(n-m)$ degrees of freedom. The number of degrees of freedom of a system is exactly equal to the number of its co-ordinates.

Thus we find that *there are always a certain number of quantities which determine the position of a rigid body or of a system of rigid bodies. They are called the co-ordinates of the body or of the system; a particular set of values of these co-ordinates determine a particular position of the body or a particular configuration of the system and slight changes in their values gives a neighbouring position or a neighbouring configuration. The body or the system is said to have a number of degrees of freedom equal to the number of its co-ordinates.*

4·3. Field of force: Conservative field: Potential Energy of a system:

If a system of forces act in a space S, so that at every point of the space there acts a definite resultant force, whose magnitude and direction is known, then such a space is called a *Field of force* or simply *a Field* and the forces of the system are called *forces of the field.*

Suppose a system of bodies connected together and subject to certain constraints, is in a certain position in a given field of force and receives a small displacement to a neighbouring position, *which does not violate any of the constraints.* Let us calculate the work done by the forces of the field which act at different points of the system.

Let Ox, Oy, Oz be a system of rect. axes chosen in the field and let X, Y, Z be the components parallel to the axes of the force P which acts at the point (x, y, z) of the system. As the field is given X, Y, Z are known functions of x, y, z. The work done by P due to the small displacement is $Xdx+Ydy+Zdz$. Therefore the total work done by the forces of the field which act on the system $=\Sigma(Xdx+Ydy+Zdz)$......(1)

Let the co-ordinates of the system be $\theta_1, \theta_2, \ldots, \theta_m$. Then x, y, z and X, Y, Z are all known functions of $\theta_1, \theta_2, \ldots, \theta_m$. Denoting the small work done by dW, we have from (1),

$$dW=f_1d\theta_1+f_2d\theta_2+\cdots+f_md\theta_m\text{......(2)}$$

where $f_1, f_2, \ldots, f_m$ are all known functions of the co-ordinates $\theta_1, \theta_2, \ldots, \theta_m$.

If their exists a function ϕ of $\theta_1, \theta_2, \ldots, \theta_m$ such that $f_1=\dfrac{\partial\phi}{\partial\theta_1}$, $f_2=\dfrac{\partial\phi}{\partial\theta_2}$, $\cdots$then we have

$$dW=\frac{\partial\phi}{\partial\theta_1}d\theta_1+\frac{\partial\phi}{\partial\theta_2}d\theta_2+\cdots+\frac{\partial\phi}{\partial\theta_m}\cdot d\theta_m=d\phi.$$

$$\therefore\ W=\phi+C, \text{ where } C \text{ is a constant.}$$

W is called the *work function.* The work done by the forces of the field when the system moves from one position A to another position $B=\int_A^B d\phi=\phi_B-\phi_A$, where ϕ_A denotes the value of ϕ for the position A i.e. $\phi_A=\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$, if $\theta_1=\alpha_1$,

$\theta_2=\alpha_2, \cdots$ define the position A. Thus the work done depends only on the initial and final positions of the system and not on the route the system has taken to move from one position to the other provided no constraints are violated. In such a case the *field of force is called conservative and the forces of the field are said to form a conservative system.*

Suppose the field is conservative and suppose there is a standard position C, defined by $\theta_1=\beta_1, \theta_2=\beta_2, \ldots$ Then the work done by the forces of the field when the system moves from its present position to the standard position is $\phi(\beta_1, \beta_2, \ldots)-\phi(\theta_1, \theta_2, \ldots)$ and this depends only on the co-ordinates of the present position. This is called the *potential energy of the system in the position* $(\theta_1, \theta_2, \ldots, \theta_m)$ and is denoted by V.

$\therefore \quad dV=-d\phi=-dW \quad \therefore \quad dW+dV=0$

or, $W+V=$constant.

Thus when the system moves in such a way that the forces of the field do *positive* work, the potential energy of the system *decreases* by the amount of work done.

It is proved in Dynamics that when a system moves from one position to another, the change in its Kinetic Energy is equal to the work done by the external forces. Therefore if K denotes the Kinetic energy of the system then $dK=dW=-dV$

$\therefore \quad dK+dV=0$

or, $K+V=$constant. Thus *in a conservative field, the sum of the Kinetic and potential energies of a system is constant* i. e. the same at all positions.

All natural fields of force are conservative ; the field due to gravity is the simplest example of a conservative field.

4·4. The Energy Test of Stability.

Suppose a body or a system of bodies, connected together and subject to certain constraints, is in equilibrium under the action of forces in a conservative field. By the principle of virtual work, no work is done by the forces of the field when the system receives any arbitrary small virtual displacement consistent with its geometrical conditions. As no work is done, there is no change in the Kinetic energy of the system and so no change in its potential energy as in a conservative field the sum of the kinetic and potential energies of a system is constant. Conversely if there is no change in the potential energy of the system for any arbitrary small displacement, then there is no change in its kinetic energy and so no work is done by the forces of the field and therefore, by the converse of the principle of virtual work, the system is in equilibrium. Thus for positions of equilibrium we have $dV=0$, where V denotes the potential energy, and therefore V is a maximum or a minimum or stationary at the positions of equilibrium.

First suppose the position of equilibrium of the system is one in which V is a maximum. Let the system be displaced to a neighbouring position and then set free. In the displaced position the value of V is decreased. When set free, it begins to move under the action of the forces of the field and therefore gains kinetic energy and so its potential energy decreases. Therefore, it cannot move back to its original position of equilibrium and the equilibrium is *unstable*.

Next suppose the position of equilibrium of the system is one in which V is a minimum and the system is displaced slightly and then set free; in the displaced position V is increased and when set free the system begins to move; and it must so move that its potential energy decreases, for by moving it is gaining kinetic energy. Therefore the system tends to move back to its original position of equilibrium, and the equilibrium is *stable*.

Lastly, when V is stationary in the position of equilibrium, it is increased for some displacements and decreased for some others. Therefore from the preceding arguments it follows that the equilibrium is stable for some displacements and unstable for others and therefore the equilibrium is on the whole unstable. Hence we have the theorem known as *the Energy test of stability*:

The positions of equilibrium of a system in a conservative field are those positions in which the potential energy of the system is a maximum or a minimum or stationary. Positions of maximum potential energy are positions of unstable equilibrium, positions of minimum potential energy are positions of stable equilibrium and the positions in which the potential energy is stationary are on the whole unstable.

In the above argument it has been assumed that no one of the neighbouring positions is also a position of equilibrium. But if there be such a neighbouring position, then the equilibrium is *neutral* for the displacement to that position.

4·5. The most important case is the one in which gravity is the only external force acting on the system.

Let a body or system of bodies, connected together and subject to certain constraints, be under the action of no forces except their weights and the forces of constraints, and these forces of constraints are supposed to be such that they do not appear in the equation of virtual work.

Let $\omega_1, \omega_2, \ldots\ldots$ be the weights of the different bodies of the system and let $z_1, z_2, \ldots\ldots$ be the depths of their centres of gravity below some fixed horizontal plane. Let W be the total weight of the system, $\bar{z}$ the depth of its c.g. below the fixed plane and V its potential energy. The virtual work of the weights for a small displacement is $-dV = \omega_1 \delta z_1 + \omega_2 \delta z_2 + \cdots\cdots$.

$$or, \quad -V + C = \omega_1 z_1 + \omega_2 z_2 + \cdots\cdots$$

$$= W\bar{z} \quad \left[\text{for } \bar{z} = \frac{\Sigma \omega z}{\Sigma \omega} \right]$$

$$or, \quad V = C - W\bar{z}, \quad \text{where } C \text{ is a constant.}$$

∴ When $\bar{z}$ is a maximum, V is a minimum and the equilibrium is stable, and when $\bar{z}$ is a minimum, V is a maximum and the equilibrium is unstable. Thus the positions of equilibrium of the system are those in which the depth of its c. g. below a fixed horizontal plane is a maximum or a minimum and the equilibrium is stable in the former case and unstable in the latter.

Let the system have one degree of freedom, then its position is determined by a single co-ordinate, say θ. From the geometry of the question, $\bar{z}$ can be expressed in terms of θ. Let $\bar{z} = f(\theta)$. The positions of equilibrium of the system are given by those values of θ which makes $\bar{z}$ a maximum or a minimum. The required values of θ are given by $\frac{d\bar{z}}{d\theta} = 0$ *or*, $f'(\theta) = 0$.

Let $\alpha, \beta, \ldots$ be the values of θ obtained by solving this equation. Find $f''(\theta)$. If $f''(\alpha) < 0$, then $\theta = \alpha$ makes $\bar{z}$ a maximum and ∴ gives a position of stable equilibrium, and if $f''(\alpha) > 0$, then $\theta = \alpha$ makes $\bar{z}$ a minimum and gives a position of unstable equilibrium. If $f''(\alpha) = 0$, we must find the higher derivatives of $\bar{z}$ and decide, according to the rules of Calculus, whether $\theta = \alpha$ makes $\bar{z}$

a maximum or a minimum. Similarly the other values of θ are to be tested for stability.

As the system moves through different positions, its c. g. will describe some definite curve, whose maximum and minimum ordinates will occur alternately. It follows that *the stable and unstable positions of equilibrium occur alternately.*

4·6. A perfectly rough heavy body rests in equilibrium on a fixed body. To find whether the equilibrium is stable or unstable.

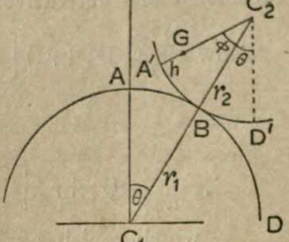
We shall discuss the problem under the following assumptions :—

(i) The very small displacements considered take place in a vertical plane of symmetry so that the centre of gravity of the upper body moves in that vertical plane during its motion.

(ii) The parts of the two bodies which are in contact during the displacements are portions of spheres. Their sections by the vertical plane of symmetry are two circular arcs and during the displacements the upper circular arc rolls on the lower one. No sliding is possible as, by assumption, the bodies are perfectly rough.

(a) We first assume that the surface of the lower body, which is fixed, is convex upwards and the upper body rests in equilibrium on the highest point of the fixed body.

Two forces acting on the upper body keep it in equilibrium viz. its weight, a vertical force through its c. g. and the total reaction at the point of contact ; it follows that the c.g. is on the vertical through the point of contact. Let h be its height above A, the point of contact in the position of equilibrium. Let r_1, r_2 be the radii of the circular arcs in which the vertical plane of symmetry cuts the lower and the upper body respectively, the two arcs touching at A.



Let $A'BD'$ be the slightly displaced position of the upper circular arc, C_2 being the new position of its centre, B the new point of contact, A' the new position of A and G the new position of the c.g. of the upper body on $A'C_2$ where $A'G=h$.

Since the upper body rolls over the lower, arc AB=arc $A'B$. Let C_1 be the centre of the lower circular arc ; then C_1BC_2 is one straight line and $C_1B=r_1$, $C_2B=A'C_2=r_2$; and $C_2G=r_2-h$ or $h-r_2$ acc. as $A'G \lessgtr A'C_2$.

Let $\angle AC_1B=\angle BC_2D'=\theta$, $\angle A'C_2B=\phi$, θ,ϕ being very small angles.

Then $r_1\theta=\text{arc } AB=\text{arc } A'B=r_2\phi$......(1)

Let z=height of G above the horizontal through C_1.

Then z=projection of C_1G on C_2A

$$=\text{sum of the projections of } C_1C_2 \text{ and } C_2G \text{ on } C_1A$$

$$=(r_1+r_2)\cos\theta-(r_2-h)\cos(\theta+\phi)$$

$$or,\quad z=(r_1+r_2)\cos\theta-(r_2-h)\cos\left(\frac{r_1+r_2}{r_2}\cdot\theta\right)\cdots[\text{from (1)}].$$

$$\therefore\quad \frac{dz}{d\theta}=-(r_1+r_2)\sin\theta+(r_2-h)\left(\frac{r_1+r_2}{r_2}\right)\sin\left(\frac{r_1+r_2}{r_2}\theta\right)$$

$$\frac{d^2z}{d\theta^2}=-(r_1+r_2)\cos\theta+(r_2-h)\left(\frac{r_1+r_2}{r_2}\right)^2\cos\left(\frac{r_1+r_2}{r_2}\theta\right) \quad\text{.....(2)}$$

In position of equilibrium $\theta=0$; $\frac{dz}{d\theta}=0$ when $\theta=0$. And the equilibrium is stable or unstable acc as the value of z when $\theta=0$ is a minimum or a maximum i.e. acc. as $\frac{d^2z}{d\theta^2}$ is $>$ or <0 when $\theta=0$ i.e. acc. as $-(r_1+r_2)+(r_2-h)\left(\frac{r_1+r_2}{r_2}\right)^2\gtrless 0$ or $\frac{1}{h}\gtrless\frac{1}{r_1}+\frac{1}{r_2}$.

If $\frac{d^2z}{d\theta^2}=0$ when $\theta=0$ i.e. if $r_1r_2=h(r_1+r_2)$, then we have to consider higher derivatives.

$$\text{Now}\quad \frac{d^2z}{d\theta^2}=(r_1+r_2)\left[-\cos\theta+\cos\left(\frac{r_1+r_2}{r_2}\theta\right)\right]$$

$$\left[\text{From (2), putting } h=\frac{r_1r_2}{r_1+r_2}.\right]$$

$$\therefore\quad \frac{d^3z}{d\theta^3}=(r_1+r_2)\left[\sin\theta-\frac{r_1+r_2}{r_2}\sin\left(\frac{r_1+r_2}{r_2}\theta\right)\right]$$

which vanishes when $\theta=0$.

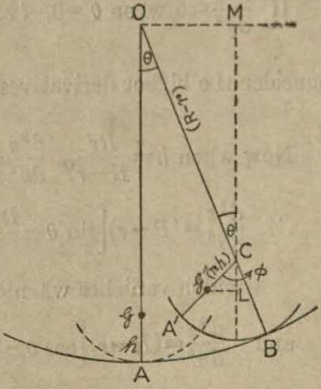
$$\text{and}\quad \frac{d^4z}{d\theta^4}=(r_1+r_2)\left[\cos\theta-\left(\frac{r_1+r_2}{r_2}\right)^2\cos\left(\frac{r_1+r_2}{r_2}\theta\right)\right]\gtrless 0,$$

when $\theta=0$, acc. as $1-\left(1+\frac{r_1}{r_2}\right)^2 \gtreqless 0 \quad \therefore \frac{d^4z}{d\theta^4}<0$, when $\theta=0$ and $\therefore$ in this case z is a max. when $\theta=0$ and so the equilibrium is unstable.

Hence the equilibrium is stable only if $\frac{1}{h}>\frac{1}{r_1}+\frac{1}{r_2}$ and is unstable in the other cases.

(*b*) Secondly, we assume that the surface of the lower fixed body is concave upwards and the upper body rests in equilibrium at the lowest point of the concave surface.

Two forces acting on the upper body keep it in equilibrium viz. its weight, a vertical force through G, its c.g. and the total reaction at the point of contact; it follows that G is on the vertical through the point of contact. Let h be the height of G above A, the point of contact in the position of equilibrium. Let r, R be the radii of the circular arcs in which the vertical plane of symmetry cuts the upper and the lower body respectively, the two arcs touching at A. Let $A'B$ be the slightly displaced position of the upper circular arc, C being the new position of its centre, B the new point of contact, A' the new position of A and G' the new position of G on $A'C$ where $A'G'=h$. Since the upper body rolls on the lower, arc $AB=$ arc $A'B$. Let O be the centre of the lower circular arc; then OCB is one straight line and $OB=R$, $CB=CA'=r<R$, and $CG'=(r-h)$ or $(h-r)$ acc. as $A'G' \lesseqgtr A'C$



Let $\angle AOB=\theta$ and $\angle A'CB=\phi$, then θ, ϕ are very small angles.

We have $R\theta=\text{arc } AB=\text{arc } A'B=r\phi \cdots\cdots(1)$

Let z be the the depth of G' below the horizontal through O, which is cut by the vertical through C at M. Draw $G'L \perp$ on MC.

Then $z=ML=MC+CL$ or, $MC-CL$ acc. as $A'G' \lesseqgtr A'C$

$$=(R-r)\cos\theta+(r-h)\cos(\phi-\theta).$$

$$=(R-r)\cos\theta+(r-h)\cos\left(\frac{R-r}{r}\theta\right), \qquad [\text{ from (1)}].$$

$$\therefore \quad \frac{dz}{d\theta} = -(R-r)\sin\theta - (r-h)\left(\frac{R-r}{r}\right)\sin\left(\frac{R-r}{r}\theta\right)$$

$$\frac{d^2z}{d\theta^2} = -(R-r)\cos\theta - (r-h)\left(\frac{R-r}{r}\right)^2\cos\left(\frac{R-r}{r}\theta\right) \cdots\cdots (2)$$

In the position of equilibrium considered, we have $\theta=0$ and $\frac{dz}{d\theta}=0$ when $\theta=0$. Therefore the equilibrium is stable or unstable acc. as z is a maximum or a minimum when $\theta=0$ *i.e.* acc. as $\frac{d^2z}{d\theta^2} \lessgtr 0$ when $\theta=0$ *i.e.* acc. as $-(R-r)-(r-h)\left(\frac{R-r}{r}\right)^2 \lessgtr 0$

$$\text{or,} \quad \frac{1}{h} \gtrless \frac{1}{r} - \frac{1}{R}$$

If $\frac{d^2z}{d\theta^2}=0$ when $\theta=0$, *i.e.* if $Rr=h(R-r)$, then we have to consider the higher derivatives $\frac{d^3z}{d\theta^3}, \frac{d^4z}{d\theta^4}$.

Now when $h=\frac{Rr}{R-r}$, $\frac{d^2z}{d\theta^2}=(R-r)\left[-\cos\theta+\cos\left(\frac{R-r}{r}\right)\theta\right]$.

$$\therefore \quad \frac{d^3z}{d\theta^3}=(R-r)\left[\sin\theta-\frac{R-r}{r}\sin\left(\frac{R-r}{r}\theta\right)\right],$$

which vanishes when $\theta=0$

and $\frac{d^4z}{d\theta^4}=(R-r)\left[\cos\theta-\left(\frac{R-r}{r}\right)^2\cos\left(\frac{R-r}{r}\right)\theta\right] \lessgtr 0$

when $\theta=0$, acc. as $1-\left(\frac{R-r}{r}\right)^2 \lessgtr 0$ or, $R \gtrless 2r$.

$\therefore$ In this case, z is a maximum or a minimum and therefore, the equilibrium is stable or unstable acc. as $R \gtrless 2r$.

When $h=\frac{Rr}{R-r}$ and $R=2r$, then $h=2r$ and we have $z=0$ for all θ even when finite,

$\therefore$ $\frac{dz}{d\theta}=0$ for all θ. So every displaced position is also a position of equilibrium and the equilibrium at $\theta=0$ is really neutral.

4·7. It may be remarked that in establishing the results in the previous article, the displacement given to the upper body is an infinitesimal and so it may be conceived to be so small that B, the new point of contact is the point consecutive to A. The two curves of intersection of the bodies by the vertical plane of symmetry touch at A which is to be considered as two consecutive points common to them and the upper one rolls to B where they touch at their next consecutive points. So when these curves of intersection are not circles, they may be replaced by their two circles of curvature at A without affecting the argument in the previous article because each circle of curvature has *three* consecutive points at A common with the respective curves of intersection. So, in this case, if ρ_1 and ρ_2 be the radii of the circles of curvature at A of the vertical sections of the upper and the lower body respectively, the equilibrium is stable or unstable acc. as $\frac{1}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}$......(1) in case ($a$) of Art 4·6 and acc. as $\frac{1}{h} \gtrless \frac{1}{\rho_1} - \frac{1}{\rho_2}$, in case ($b$) of same article.

When $\frac{1}{h} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$, the equilibrium is said to be *neutral to a first approximation.* Determination of stability in this case is difficult.

If one of the bodies is a plane, the curvature of its section is zero and the condition of stability is obtained by putting $\frac{1}{\rho_1}$ or $\frac{1}{\rho_2}$ equal to zero in (1) acc. as the plane body is the upper or the lower one. If the upper body rests with its plane face on the lower body, the equilibrium is stable or unstable acc. as $\frac{1}{h} \gtrless \frac{1}{\rho_2}$, and if the upper body rests on a plane face of the lower body, the equilibrium is stable or unstable acc. as $\frac{1}{h} \gtrless \frac{1}{\rho_1}$.

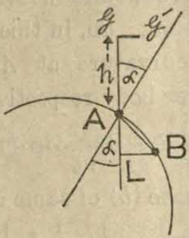
4·8. Rocking Stones.

In Art. 4·6 we discussed the stability of equilibrium of one body resting on another on the assumptions that the portions of the two bodies in contact are spherical and that their point of contact is the highest point of the lower body. Now we remove both these restrictions in discussing the stability.

A perfectly rough heavy body rests in equilibrium on a fixed body at the point A. The common normal at the point of contact

A makes an angle α *with the vertical. The c. g. of the upper body, G, which for equilibrium must lie on the vertical at A, is at the height h above A. The equilibrium is stable or unstable acc. as* $\frac{\cos\alpha}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}$, *where* ρ_1, ρ_2 *are the radii of curvatures at A of the curves of intersection in which the two bodies are cut by the vertical plane of symmetry in which the displacement is given.*

Let the upper body roll on the lower through a small angle θ so that the new point of contact is B and let arc $AB = s$. Then we have, as in article 3·10(e), $\theta = s\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$, neglecting small quantities of second order where θ is a small quantity of first order. Since A is the instantaneous centre of rotation of the upper body when it rolls, G moves horizontally to G' and its horizontal displacement GG' is $h\theta$.



Draw $BL \perp$ to the vertical AG. Then the equilibrium is stable or unstable acc. as $GG' \lesseqgtr BL$. For when $GG' < BL$ the line of action of the weight of the upper body, which is the vertical through G', passes to the left of B and the moment of this weight about B will tend to move the upper body back to its original position of equilibrium. The reverse will happen in the other case. Now from $\triangle BAL$, $BL = AB \sin BAL = s \cos\alpha$, neglecting small quantities of higher order [for when B is very near to A, AB becomes the tangent at A and $\angle BAL$ becomes $(90° - \alpha)$].

Hence the equilibrium is stable or unstable according as

$$hs\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \lesseqgtr s.\cos\alpha \quad \text{or} \quad \text{acc. as. } \frac{\cos\alpha}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

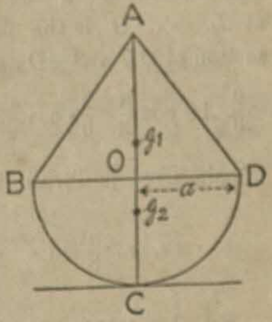
In particular, if the upper body rests at the highest point of the lower, we have $\alpha = 0$ and the condition of stability or unstability becomes $\frac{1}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}$, the result deduced in Art. 4·7.

4·9. Problems.

In most of the problems, gravity is the only external force and the system has one degree of freedom. Method of procedure to be adopted for solving such problems has been indicated in Art. 4·5. In some problems results proved in Art. 4·6 and the more general result proved in Art. 4·8 are directly applicable. Some problems of different types will now be solved for purposes of illustration.

Examples Worked

Ex. 1. A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table; show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.



Let ABD be the cone and BCD the hemisphere. Let AOC be the common axis on which lie the c.g.'s G_1, G_2 of the cone and the hemisphere $\therefore$ G, the c.g. of the combined body is on AOC. If G does not coincide with O, the centre of the sphere, then in position of equilibrium AOC is vertical and C is the point of contact with the horizontal table.
Let a = radius of hemisphere and h = height of cone.

$$\therefore \quad CG_2 = \frac{5}{8}a \text{ and } CG_1 = a + \frac{h}{4}.$$

volume of hemisphere $= \frac{2}{3}\pi a^3$, volume of cone $= \frac{1}{3}\pi a^2 h$.

$$\therefore \; CG = \frac{\frac{2}{3}\pi a^3 \times \frac{5}{8}a + \frac{1}{3}\pi a^2 h\left(a + \frac{h}{4}\right)}{\frac{2}{3}\pi a^3 + \frac{1}{3}\pi a^2 h} = \frac{5a^2 + 4ah + h^2}{4(2a+h)}$$

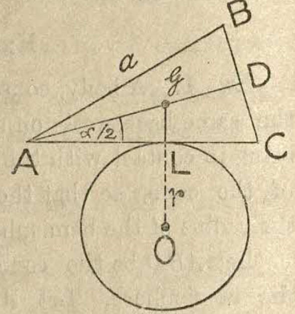
Equilibrium is stable if $\frac{1}{CG} > \frac{1}{a}$ *or*, $\frac{4(2a+h)}{5a^2+4ah+h^2} > \frac{1}{a}$

or, $3a^2 > h^2$ *or*, $h < a\sqrt{3}$.

$\therefore$ Greatest value of h for stable equilibrium $= a\sqrt{3}$.

If G coincides with O, which happens when $h = a\sqrt{3}$, the system will be in equilibrium with any point of BCD in contact with the table.

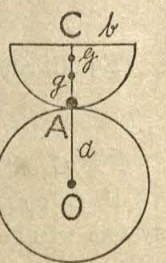
Ex. 2. A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < \frac{3r}{a}$, where a is one of the equal sides of the triangle.



Let ABC be the triangle where $AC=AB=a$ and $\angle BAC=\alpha$. The triangle rests at the highest point L of the sphere with side AC in contact. Let $BD=DC$, then the c.g. G of the triangle is on AD, and also it must be on the vertical OL at L. $\therefore$ G is the point of intersection of OL and AD and $AG=\frac{2}{3}AD$ $=\frac{2}{3}\cdot AC\cos\frac{\alpha}{2}=\frac{2}{3}a\cos\frac{\alpha}{2}$, and $h=LG=AG\sin\frac{\alpha}{2}=\frac{2}{3}a\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ $=\frac{a}{3}\sin\alpha$. Equilibrium is stable if $\frac{1}{h}>\frac{1}{r}$

$$or,\ r>\frac{a}{3}\sin\alpha \quad or,\ \sin\alpha<\frac{3r}{a}.$$

Ex. 3. A thin hemispherical bowl, of radius b and weight W, rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight ω. Show that the equilibrium is not stable unless $\omega<W.\ \frac{a-b}{2b}$.



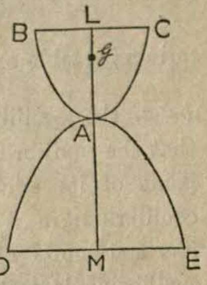
Let A be the highest point of the fixed sphere of centre O and let C be the centre of the plane face of the bowl. Then OAC is the common normal at A and it is vertical. In position of equilibrium, g, the c. g. of the bowl, must lie on the vertical at A. $\therefore$ $OAgC$ is one straight line and A is the lowest point of the bowl ; we have, $Cg=\frac{b}{2}$. The weight ω always occupies the lowest position in the bowl and its line of action always passes through C. $\therefore$ Its effect will remain unaltered if it be fixed to the bowl at C. If G be the c.g. of the combined body, the bowl and the weight ω fixed at C, then $h = AG = \dfrac{W.\ \frac{b}{2} + \omega.\ b}{W+\omega}$. For stable equilibrium, $\dfrac{1}{h} > \dfrac{1}{a} + \dfrac{1}{b}$

$$or,\quad \frac{2(W+\omega)}{b(W+2\omega)} > \frac{a+b}{ab}$$

$$or,\quad 2a(W+\omega) > (a+b)(W+2\omega)$$

$$or,\quad \omega < \frac{W.(a-b)}{2b}.$$

Ex. 4. A solid frustum of a paraboloid of revolution, of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$; show that the equilibrium is stable if $h > \dfrac{3ab}{a+b}$. [C. H. 1951]



Let the paraboloids DAE, BAC touch at their common vertex A, where LAM is their common axis. Then centre of gravity G of the upper one BAC is on AL, where $AG = \frac{2}{3}AL = \frac{2}{3}h$. For the paraboloid can be broken up into thin circular discs by planes $\perp$ to its axis AL. Vol. of such a disc at a distance x from $A = \pi y^2 dx = \pi.4ax dx$ and its c.g. is on AL at the distance x from A $\therefore$ $AG = \dfrac{\int_0^h \pi.4axdx.x}{\int_0^h \pi.4axdx} = \frac{2}{3}h.$

Sections of the paraboloids by any vertical plane of symmetry in which the displacement is given, are the generating parabolas of latus rectum $4a$ and $4b$ respectively, A being their common vertex. We know that the radius of curvatnre at the vertex of a parabola is equal to its semi-latus rectum $\therefore$ Here $\rho_1 = 2a$, and $\rho_2 = 2b$.

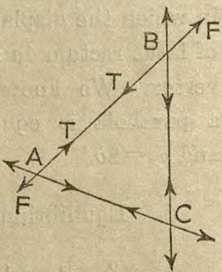
$$\therefore\ \text{Equilibrium is stable if } \frac{1}{AG} > \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

$$or,\quad \frac{3}{2h} > \frac{1}{2a} + \frac{1}{2b} \quad or,\quad h < \frac{3ab}{a+b}.$$

Ex. 5. A heavy vertical elliptic lamina, capable of moving in its own plane, rests in equilibrium on a smooth horizontal plane. Find the positions of equilibrium and determine whether they are stable or unstable. [C. H. 1952]

The centre C of the lamina is its c.g. Let A be its point of contact with horizontal plane when in equilibrium. Since plane is smooth the reaction at A is normal which, being perpendicular to horizontal plane, is vertical. It balances the weight which acts at C. $\therefore$ The normal at A passes through C. If A be $(a \cos \theta, b \sin \theta)$ the equation of normal at A is $ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta$. If it passes through C $(0, 0)$ then $\sin \theta \cos \theta = 0$ $\therefore$ $\theta = 0$, π or $\frac{\pi}{2}, \frac{3\pi}{2}$. $\therefore$ A is an end point of the major axis or of the minor axis. Let A be an end point of the major axis ; then $h = AC = a$ and $\rho = \frac{a^2 b^2}{p^3} = \frac{a^2 b^2}{a^3} = \frac{b^2}{a}$. $\therefore$ Equilibrium is stable or unstable acc. as $\frac{1}{h} \gtreqless \frac{1}{\rho}$ or $\frac{b^2}{a} \gtreqless a$ or $b \gtreqless a$. As $b < a$, the equilibrium is unstable. Similarly it can be shown that the equilibrium is stable when the lamina rests on an end point of its minor axis. Hence there are four positions of equilibrium in which the lamina touches the plane at one of the four end points of its axes. At the end points of the major axis, equilibrium is unstable but is stable at the end points of the minor axis.

Ex. 6. Three equal particles repelling each other with forces proportional to the nth. power of the distance are connected together by three equal elastic strings. Find the position of equilibrium and show that it is stable if $n < \frac{p}{p-a}$, where a is the unstretched and p the stretched length of any string.



Let A, B, C be the three equal particles and let λ be the modulus of elasticity of the equal strings. The force of repulsion on B due to A is $\mu.(AB)^n$. Let $BC = a + x$, $CA = a + y$ and $AB = a + z$. Then if F be the force of repulsion between A and B and T the tension of string AB, the work done by the two forces of repulsion and the two tensions when AB gets a small displacement

$$=Fd(AB)-Td(AB)=(F-T)dz$$
$$=\left[\mu(a+z)^n-\lambda.\frac{z}{a}\right]dz$$

$\therefore$ when the system of the 3 particles gets a small displacement the work done by the forces acting on the system is eqnal to $dW=\Sigma\left[\mu(a+x)^n-\frac{\lambda}{a}x\right]dx.$

But $dW=-dV=-\left(\frac{\partial V}{dx}dx+\frac{\partial V}{dy}dy+\frac{\partial V}{dz}dz\right)$

$\therefore$ $\frac{\partial V}{dx}=-\mu(a+x)^n+\frac{\lambda}{a}x$, and similar values of $\frac{\partial V}{dy}$ and $\frac{\partial V}{dz}$.

For stationary values of V we have $\frac{\partial V}{dx}=0, \frac{\partial V}{dy}=0, \frac{\partial V}{dz}=0$

$\therefore$ $\mu(a+x)^n=\frac{\lambda}{a}x$, $\mu(a+y)^n=\frac{\lambda}{a}y$, and $\mu(a+z)^n=\frac{\lambda}{a}z$.

$\therefore$ $x=y=z$ each being given by the same equation, $\mu t^n =\frac{\lambda}{a}(t-a)$, writing $a+x=t$. Thus in position of equilibrium the particles from an equilateral triangle whose side is given by the equation $a\mu t^n=\lambda(t-a)$.

Let p be the root of this eqn. then $a\mu p^n=\lambda(p-a)$......(1).

To test for stability we have to determine whether V is max. or min. for $x=y=z=p-a$.

Now $dV=\Sigma\left[\frac{\lambda x}{a}-\mu(a+x)^n\right]dx$

$\therefore$ $d^2V=\Sigma\left[\frac{\lambda}{a}-\mu n(a+x)^{n-1}\right](dx)^2$.

When $x=y=z=p-a$, $d^2V=\left(\frac{\lambda}{a}-\mu np^{n-1}\right)(dx^2+dy^2+dz^2)$

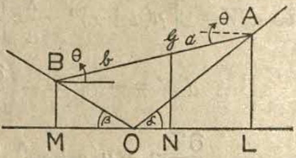
$\therefore$ V is min. or max. acc. as $\frac{\lambda}{a}-\mu np^{n-1}\gtrless 0$

or, using (1) acc. as $\lambda p-n\lambda(p-a)\gtrless 0$ or, $n\lessgtr\frac{p}{p-a}$.

Thus equilibrium is stable if $n<\frac{p}{p-a}$, where in equilibrium length of each string is p given by equation (1).

Ex. 7. A heavy rod rests with its ends on two smooth inclined planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β and the centre of gravity of the rod divides it in the ratio $a : b$, find the position of equilibrium of the rod and show that it is unstable.

Let AB be the rod, l its length, and G its c g. Then $AG : GB = a : b$. Let AL, BM, GN be drawn $\perp^s$ on the horizontal. For equilibrium, the three forces which act on the rod viz. the two normal reactions at A, B and the wt. of the rod must lie in the same plane and meet in a point. This plane is vertical and cuts the inclined planes along lines of greatest slope AO and BO. AL, BM, GN all lie in this vertical plane and $MONL$ is a horizontal line and $\angle AOL=\alpha$ and $\angle BOM=\beta$. If z be the height of G above the line $MONL$, we have $z=GN=\dfrac{(b.AL+a.BM)}{(a+b)}$, where $AL= OA \sin\alpha$ and $BM=OB\sin\beta$. Now if θ be the inclination of AB to the horizontal, then $\angle OBA=\beta+\theta$, $\angle OAB=(\alpha-\theta)$ and $\angle AOB =\pi-(\alpha+\beta)$. $\therefore$ from $\triangle OAB$, we have,



$$\frac{OA}{\sin(\beta+\theta)}=\frac{OB}{\sin(\alpha-\theta)}=\frac{l}{\sin(\alpha+\beta)}$$

$$\therefore\quad (a+b)z=bl.\ \frac{\sin\alpha}{\sin(\alpha+\beta)}.\sin\ (\beta+\theta)+\frac{la\sin\beta}{\sin(\alpha+\beta)}\sin(\alpha-\theta)$$

$$\therefore\quad (a+b)\frac{dz}{d\theta}=\frac{l}{\sin(\alpha+\beta)}\left[b\sin\alpha\cos(\beta+\theta)-a\sin\beta.\cos(\alpha-\theta)\right]$$

$$\text{and}\quad (a+b)\frac{d^2z}{d\theta^2}=\frac{l}{\sin(\alpha+\beta)}\left[-b\sin\alpha\sin(\beta+\theta)-a\sin\beta.\sin(\alpha-\theta)\right]$$

For equilibrium, we have, $\dfrac{dz}{d\theta}=0$

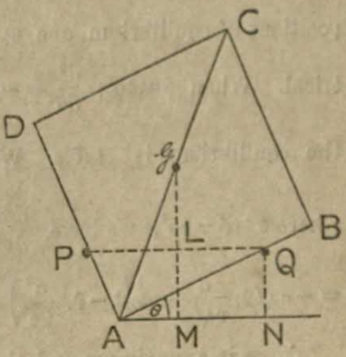
$$\therefore\quad b\sin\alpha\cos(\beta+\theta)-a\sin\beta\cos(\alpha-\theta)=0$$

or, $\tan\theta=\dfrac{b\cot\beta-a\cot\alpha}{a+b}$, which gives the position of equilibrium of the rod.

$$\text{And, } (a+b)\,\frac{d^2z}{d\theta^2}=\frac{-l\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}$$
$$[(a+b)+\tan\theta(b\cot\beta-a\cot\alpha)]$$

When $\tan\theta=\frac{b\cot\beta-a\cot\alpha}{a+b}$, $\frac{d^2z}{d\theta^2}<0$, as θ being acute, $\cos\theta>0$ and expression within square bracket is $+ve$. $\therefore$ z is a maximum and therefore the position of equilibrium is *unstable.*

Ex. 8. A uniform square lamina rests in equilibrium in a vertical plane with two of its sides in contact with smooth pegs in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by $c(\sin\theta+\cos\theta)=a$, $2a$ being the length of a side of the square. Discuss the stability of possible positions of equilibrium. [C. H. 1955]



Let $ABCD$ be the square and P, Q the pegs. Then $PQ=c$. Let side AB make an angle θ with the horizontal AMN. G, the c.g. of the lamina is the mid-point of AC. Let GLM be drawn $\perp$ to AMN cutting PQ at L and AMN at M and let z be the height of G above fixed line PQ. We have,

$$z=GL=GM-LM=a\sqrt{2}.\sin(\theta+45^\circ)-AQ\sin\theta$$
$$=a(\cos\theta+\sin\theta)-c\cos\theta.\sin\theta.$$

$$\therefore\quad \frac{dz}{d\theta}=a(-\sin\theta+\cos\theta)-c(\cos^2\theta-\sin^2\theta)$$

$$\frac{d^2z}{d\theta^2}=a(-\cos\theta-\sin\theta)-c(-4\sin\theta\cos\theta)$$

For positions of equilibrium, we have $\frac{dz}{d\theta}=0$,

or, $(\cos\theta-\sin\theta)[a-c(\cos\theta+\sin\theta)]=0$

$\therefore$ $\tan\theta=1$ *or* $c(\cos\theta+\sin\theta)=a.$

Thus the positions of equilibrium are given by $\theta=45^\circ$ and $c(\cos\theta+\sin\theta)=a$; when $\theta=45^\circ$, AC is vertical; it is the symmetrical position of equilibrium. The non-symmetrical positions of equilibrium are given by the equation $c(\cos\theta+\sin\theta)$ $=a$ *or* $\sin(\theta+45^\circ)=\frac{a}{c\sqrt{2}}$ and θ is real when $a<c\sqrt{2}$ *or* $c>\frac{a}{\sqrt{2}}=\frac{1}{4}AC$. As $\theta>0$, we must also have $\frac{a}{c\sqrt{2}}>\frac{1}{\sqrt{2}}$ *or* $c<a$.

Let $a>c>\frac{1}{4}AC$ and write $\phi=\theta+45^\circ$. $\therefore$ there are also two non-symmetrical positions of equilibrium given by $\sin\phi=\frac{a}{c\sqrt{2}}$. When $c<\frac{1}{4}AC$, the symmetrical position is the only position of equilibrium.

For stability : (i) Let $a>c>\frac{1}{4}AC$ $\therefore$ $c>\frac{a}{\sqrt{2}}$. There are 3 positions of equilibrium, one symmetrical and two non-symmetrical. When $\theta=45^\circ$, $\frac{d^2z}{d\theta^2}=-a\sqrt{2}+2c>0$, z is a min and the equilibrium is *stable*. When θ is given by $\sin\phi=\frac{a}{c\sqrt{2}}$ where $\theta=\phi-45^\circ$, we have, $\frac{d^2z}{d\theta^2}=-a\sqrt{2}.\sin\phi+2c\sin(2\phi-90^\circ)$

$$=-a\sqrt{2}.\frac{a}{c\sqrt{2}}-2c\left(1-2.\frac{a^2}{2c^2}\right)=-\frac{a^2}{c}-2c+\frac{2a^2}{c}=\frac{a^2-2c^2}{c}<0$$

$\therefore$ z is a maximum and the equilibrium is *unstable.* Thus the symmetrical position is stable and the two non-symmetrical positions are unstable.

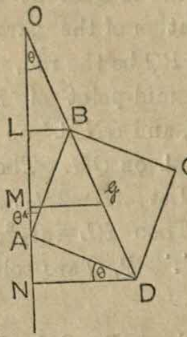
(ii) Let $c<\frac{1}{4}AC$ *i.e.* $c<\frac{a}{\sqrt{2}}$. There is only one position of equilibrium, the symmetrical one.

When $\theta=45^\circ$, $\frac{d^2z}{d\theta^2}=-a\sqrt{2}+2c<0$ $\therefore$ z is a maximum and the equilibrium is unstable.

Ex. 9. A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.

[C. H. 1964, 1957]

Let $ABCD$ be the square and OB the string connecting corner B to the point O in the wall. The lamina rests against the wall at A. Let $AB=a$. Then $OB=a$; let $\angle BOA=\angle BAO=\theta$. The c. g. of the lamina is G, the mid-point of BD. Draw BL, GM, DN perpendiculars on the wall. We have, $LA=a\cos\theta$, $AN=a\sin\theta$. $\therefore LN=a\cos\theta+a\sin\theta$ $\therefore LM=\frac{1}{2}LN=\frac{1}{2}a(\cos\theta+\sin\theta)$.



If z be the depth of G below fixed point O, we have, $z=OM=OL+LM$

$$=a\cos\theta+\tfrac{1}{2}a(\cos\theta+\sin\theta)$$

$$=\frac{a}{2}(3\cos\theta+\sin\theta).$$

$$\therefore\quad \frac{dz}{d\theta}=\frac{a}{2}(-3\sin\theta+\cos\theta)$$

$$\frac{d^2z}{d\theta^2}=\frac{a}{2}(-3\cos\theta-\sin\theta).$$

For positions of equilibrium, $\frac{dz}{d\theta}=0$

$\therefore$ $3\sin\theta=\cos\theta$ or, $\tan\theta=\frac{1}{3}$.

$\therefore$ In position of equilibrium, the string makes an angle $\alpha=\tan^{-1}(\frac{1}{3})$ with the vertical. As this angle α is acute, $\cos\alpha$, $\sin\alpha$ are both >0.

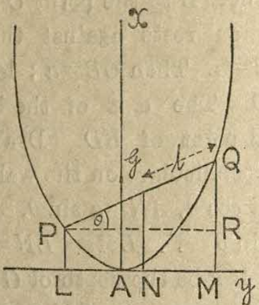
$\therefore$ When $\theta=\alpha$, $\frac{d^2z}{d\theta^2}=-\frac{a}{2}(3\cos\alpha+\sin\alpha)<0$ $\therefore$ z is a max.

Hence the position of equilibrium is stable.

Ex. 10. A uniform rod, of length $2l$, is attached by smooth rings at both ends to a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium is given by $\cos^2\theta=\frac{2a}{l}$; and that, if these positions exist, they are stable.

Show also that the positions in which the rod is horizontal are stable or unstable according as the rod is below or above the focus. [C. H. 1967]

Let A be the vertex of the parabola and Ax its axis and Ay the tangent at the vertex. The equation of the parabola is $y^2=4ax$. Let PQ be the rod, and G its c. g., the mid-point of PQ. Draw PL, QM, and GN $\perp^s$ on Ay; and draw PR $\perp$ on QM. Then $\angle QPR=\theta$.



Let $AL=2at_1$, $AM=2at_2$.

Then $PL=at_1{}^2$, $QM=at_2{}^2$

$\because$ P, Q are points on the parabola

$\therefore$ $QR=QM-PL=a(t_2{}^2-t_1{}^2)$

or, $2l \sin\theta=a(t_2{}^2-t_1{}^2)$......(1)

And $LM=AL+AM=2a(t_1+t_2)$

$\therefore$ $2l\cos\theta=2a(t_1+t_2)$ ....(2)

If z be the height of G above Ay, we have

$z=GN=\frac{1}{2}(PL+QM)=\frac{1}{2}a(t_1{}^2+t_2{}^2)$.

Now we have to express z in terms of θ with the help of the relations (1) and (2). From (1) and (2) by division, we get, $2\tan\theta=t_2-t_1$......(3)

$$\therefore\quad z=\frac{a}{4}\left[(t_1+t_2)^2+(t_1-t_2)^2\right]$$

$$=\frac{a}{4}\left[\frac{l^2}{a^2}\cos^2\theta+4\tan^2\theta\right]\text{[from (2) and (3).]}$$

$$\therefore\quad \frac{dz}{d\theta}=\frac{a}{4}\left[-2\frac{l^2}{a^2}\cos\theta\sin\theta+8\tan\theta.\sec^2\theta\right].$$

$$=\frac{a}{2}\sin\theta\left[4\sec^3\theta-\frac{l^2}{a^2}\cos\theta\right]$$

$$\frac{d^2z}{d\theta^2}=\frac{a}{2}\left[\cos\theta\left(4\sec^3\theta-\frac{l^2}{a^2}\cos\theta\right)+\sin\theta\left(12\sec^3\theta\tan\theta+\frac{l^2}{a^2}\sin\theta\right)\right]$$

......(4).

For positions of equilibrium, $\frac{dz}{d\theta}=0$,

$\therefore$ $\sin\theta=0$ *or*, $4-\frac{l^2}{a^2}\cos^4\theta=0$ *or*, $\cos^2\theta=\frac{2a}{l}$.

$\therefore$ $\theta=0$; it gives the horizontal position of equilibrium.

If $2a>l$ *or*, $2l<4a$ *i.e.* $PQ<$latus rectum, then θ is not real. ∴ there is no other position of equilibrium and the horizontal position is the only position of equilibrium. In this case, when $\theta=0$, we have from (4),

$$\frac{d^2z}{d\theta^2}=\frac{a}{2}\left(4-\frac{l^2}{a^2}\right)=\frac{1}{2a}\,(2a+l)(2a-l)>0.$$

∴ z is a min. and ∴ the horizontal position of equilibrium is stable. As $PQ<$ latus rectum, PQ is below focus.

If $2a<l$, then θ is real. Let $\cos\alpha=\sqrt{\frac{2a}{l}}$ where $0<\alpha<90°$.

Then $\cos\theta=\pm\cos\alpha$ ∴ $\theta=\alpha$ *or*, $\pi-\alpha$.

In this case, there are two other inclined positions of equilibrium in which the rod is inclined to the horizon at angles α and $(180°-\alpha)$ where $\cos\alpha=\sqrt{\frac{2a}{l}}$. Now to test for stability of these equilibrium positions :

Horizontal position—

We have $\theta=0$, ∴ $\frac{d^2z}{d\theta^2}=\frac{1}{2a}\,(2a+l)(2a-l)<0$ ∴ z is a max. and the equilibrium is *unstable*. As $PQ>$latus rectum, PQ is above focus.

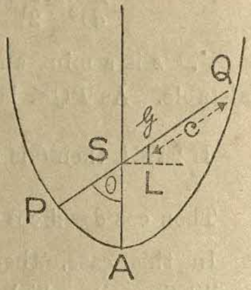
Inclined positions :

Here $\theta=\alpha$ *or*, $\pi-\alpha$, and $\cos\alpha=\sqrt{\frac{2a}{l}}$

We have, $\frac{d^2z}{d\theta^2}=\frac{a}{2}\sin^2\alpha\left(12\sec^4\alpha+\frac{l^2}{a^2}\right)>0$ ∴ z is a min. and the equilibrium is stable in both the positions.

Hence if the rod > the latus rectum, there are 3 positions of equilibrium ; one horizontal, which is unstable, and two others inclined, both stable and if the rod<the latus rectum, there is only one position of equilibrium which is horizontal and stable.

Ex. 11. A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation $\cos^4\frac{\theta}{2}=\frac{a}{2c}$, where $4a$ is the latus rectum and $2c$ the length of the rod. Investigate also the stability of the equilibrium in this position. [I. C. S. 1939]



Let A be the vertex and S the focus of the parabola, the axis AS being vertical. Let G be the middle point of the rod $PQ=2c$, which passes through S and makes an angle θ with SA. Draw $GL \perp$ on horizontal through S. Polar equation of the parabola, referred to S as pole and SA as initial line is

$$\frac{2a}{r}=1+\cos\theta.$$

$$\therefore\quad SP=\frac{2a}{1+\cos\theta} \text{ and } SG=PG-PS=c-PS$$

$\therefore$ If z be the height of G above the fixed point S, we have,

$$z=GL=SG\cos\theta=\cos\theta.\left[c-\frac{2a}{1+\cos\theta}\right]$$

$$=\cos\theta\left(c-a\sec^2\frac{\theta}{2}\right)$$

$$\therefore\quad \frac{dz}{d\theta}=-\sin\theta\left(c-a\sec^2\frac{\theta}{2}\right)-\cos\theta.\,a.\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}$$

$$=-c\sin\theta+a\sec^3\frac{\theta}{2}\sin\frac{\theta}{2}=\sin\theta\left(-c+\frac{a}{2}\sec^4\frac{\theta}{2}\right)$$

$$\frac{d^2z}{d\theta^2}=\cos\theta\left(-c+\frac{a}{2}\sec^4\frac{\theta}{2}\right)+\sin\theta\left(a\sec^4\frac{\theta}{2}\tan\frac{\theta}{2}\right)$$

For positions of equilibrium,

$$\frac{dz}{d\theta}=0\quad\therefore\quad \theta=0\quad or,\ \cos^4\frac{\theta}{2}=\frac{a}{2c}.$$

$\theta=0$ gives the vertical position of equilibrium. The other positions of equilibrium are given by $\cos^4\frac{\theta}{2}=\frac{a}{2c}$, where θ is the inclination of the rod to the vertical, θ being real as $a<2c$.

Let $\left(\frac{a}{2c}\right)^{\frac{1}{4}}=\cos\alpha(0<\alpha<90°)$, then $\cos\frac{\theta}{2}=\cos\alpha$.

$\therefore\ \dfrac{\theta}{2} = \pm\alpha$ *or,* $\theta = \pm 2\alpha$. There are two inclined positions of equilibrium in which the rod makes equal angles with the axis but on opposite sides of it.

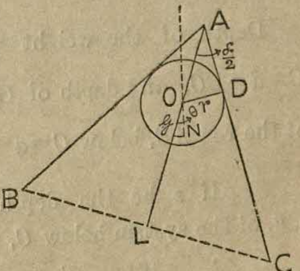
When $\theta = \pm 2\alpha$, $\dfrac{d^2z}{d\theta^2} = \sin 2\alpha . a \sec^4\alpha \tan\alpha = 2a \sin^2\alpha \sec^4\alpha > 0$.

$\therefore$ z is a min. and $\therefore$ the inclined positions are positions of *stable* equilibrium.

When $\theta = 0$, $\dfrac{d^2z}{d\theta^2} = \frac{1}{2}(a - 2c) < 0$ $\therefore$ z is a max. and $\therefore$ the vertical position is unstable.

Ex. 12. Two equal uniform rods are firmly jointed at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in stable or unstable equilibrium according as the length of either rod is $\gtrless 4r \operatorname{cosec} \alpha$.

Let AB, AC be the two rods firmly jointed at A, so that $\angle BAC = \alpha$, and which rest on a sphere of radius r. Let G be the c. g. of the system. It is on AL, which is drawn $\perp$ on BC, and $AG = \frac{1}{2}AL = \frac{1}{2}l \cos\dfrac{\alpha}{2}$, where l = length of a rod. Let AL make an angle θ with the vertical ON and let z be the depth of G below the centre O of the fixed sphere.



$$\therefore\ z = ON = OG \cos\theta = (AG - AO)\cos\theta$$

$$= \left(\frac{1}{2}l \cos\frac{\alpha}{2} - r \operatorname{cosec}\frac{\alpha}{2}\right) \cdot \cos\theta = k\cos\theta, \text{ say}$$

$$\therefore\ \frac{dz}{d\theta} = -k\sin\theta,\ \frac{d^2z}{d\theta^2} = -k\cos\theta.$$

For positions of equilibrium, $\dfrac{dz}{d\theta} = 0$ $\therefore$ $-k\sin\theta = 0$ *or* $\theta = 0$.

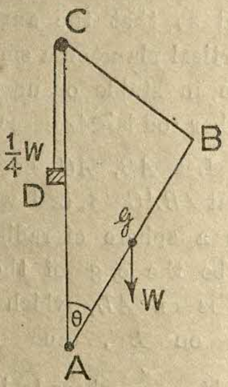
When $\theta = 0$, $\dfrac{d^2z}{d\theta^2} = -k \lessgtr 0$, acc. as $k \gtrless 0$.

or, acc. as $l \cos \frac{\alpha}{2} \gtreqless 2r \operatorname{cosec} \frac{\alpha}{2}$

or, acc. as $l \gtreqless 4r \operatorname{cosec} \alpha$.

$\therefore$ z is a max. or a min. and therefore the equilibrium is stable or unstable acc. as $l \gtreqless 4r \operatorname{cosec} \alpha$.

Ex. 13. One end A of a uniform rod AB, of weight W and length l, is smoothly hinged to a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A and carries a weight $\frac{1}{4}W$. If $l<a<2l$, show that the system is in stable equilibrium when AB is vertically upwards and there is also a configuration of equilibrium in which the rod is inclined at a certain angle to the vertical.



Let the rod AB make an angle θ with AC where C is the pulley and $AC=a$. And let d be the length of the string.

Now $BC=\sqrt{l^2+a^2-2la \cos \theta}$.

Depth of the weight $\frac{W}{4}$ below $C=d-BC$, and depth of G, the c. g. of the rod AB, below $C=a-\frac{l}{2}\cos\theta$.

$\therefore$ If z be the depth of the c. g. of the system below C, we have,

$$\left(W+\frac{W}{4}\right)z=\frac{W}{4}(d-BC)+W\left(a-\frac{l}{2}\cos\theta\right)$$

or, $5z=d-\sqrt{l^2+a^2-2la\cos\theta}+4a-2l\cos\theta$.

$$\therefore\quad 5\frac{dz}{d\theta}=-\frac{al\sin\theta}{\sqrt{l^2+a^2-2la\cos\theta}}+2l\sin\theta.$$

$$5\frac{d^2z}{d\theta^2}=-al\left[\frac{\cos\theta}{\sqrt{l^2+a^2-2al\cos\theta}}-\frac{al\sin^2\theta}{(l^2+a^2-2al\cos\theta)^{\frac{3}{2}}}\right]+2l\cos\theta.$$

For positions of equilibrium, $\frac{dz}{d\theta}=0$

$$\therefore\quad l\sin\theta\left[2-\frac{a}{\sqrt{l^2+a^2-2la\cos\theta}}\right]=0.$$

$\sin\theta=0$ gives $\theta=0,\ \pi$, the vertical positions of equilibrium. The other position of equilibrium is given by

$$4(l^2+a^2-2al\cos\theta)=a^2 \quad or, \quad \cos\theta=\frac{4l^2+3a^2}{8al}.$$

This will give a real value of θ if $4l^2+3a^2<8al$

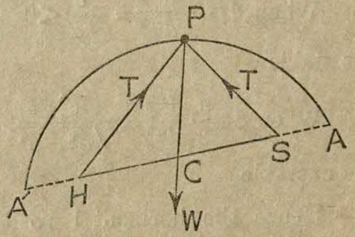
or, $(2l-a)(2l-3a)<0$, which holds as it is given that $l<a<2l$. Therefore there is another position of equilibrium besides the vertical positions. Now to test the stability of equilibrium in the vertical position when AB is upwards, we have,

$$\text{when } \theta=0,\ 5\frac{d^2z}{d\theta^2}=2l-\frac{al}{|a-l|}=2l-\frac{al}{a-l}=\frac{l(a-2l)}{(a-l)}<0$$

$\therefore$ z is a max. and therefore the equilibrium position in which the rod is vertically upwards is stable.

Ex. 14. A uniform rod is hung from a smooth peg by means of a light inextensible string attached to its ends and passing over the peg. Show that the rod is unstable when it is horizontal. When is the rod stable ? [C. H. 1968, 1958]

Let HS be the rod of length $2c$ and let the length of the string be $2a$. Let P be the peg. Since $HP+SP=$length of the string$=2a$, a constant, all possible positions of the peg lie on an ellipse whose foci are H and S and whose major axis is $2a$ Let AA' be the major-axis, and C the centre. Then $HC=CS=c$



$\therefore$ If e be the eccentricity of the ellipse, we have $c=ae$.

$\therefore$ $b^2=a^2-a^2e^2=a^2-c^2\cdots\cdots(1)$, $2b$ being the minor axis. Since the rod is in equilibrium under the action of the two equal tensions along HP, SP, which act at P and the vertical weight of the rod which acts at C it follows that in positions of equilibrium CP is vertical. Draw the ellipse APA' and take any point P on the ellipse as the position of the peg and join CP. Then hang the figure from the peg so that P is at the peg and PC is vertical. If PC is a max. or a min. then the system will be in equilibrium when placed in the position of the figure. Take C as origin and CA as the x-axis, and let P be $(a\cos\theta, b\sin\theta)$. Then $CP=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$

$$=\sqrt{a^2-c^2\sin^2\theta}\quad\text{[from (1).]}$$

If z be the depth of C below fixed point P, we have,

$$z=\sqrt{a^2-c^2\sin^2\theta}$$

$$\therefore\quad \frac{dz}{d\theta}=\frac{-c^2\sin\theta\cos\theta}{\sqrt{a^2-c^2\sin^2\theta}}.$$

$$\frac{d^2z}{d\theta^2}=-c^2\left[\frac{\cos^2\theta-\sin^2\theta}{\sqrt{a^2-c^2\sin^2\theta}}+\frac{c^2\sin^2\theta\cdot\cos^2\theta\cdot}{(a^2-c^2\sin^2\theta)^{\frac{3}{2}}}\right].$$

For positions of equilibrium, $\frac{dz}{d\theta}=0$

$$\therefore\quad \sin\theta=0 \quad or \quad \cos\theta=0\,; \quad or \quad \theta=0,\ \frac{\pi}{2}.$$

When $\theta=0$, rod is vertical and when $\theta=\frac{\pi}{2}$, rod is horizontal. Thus there are two positions of equilibrium in which the rod is horizontal or vertical.

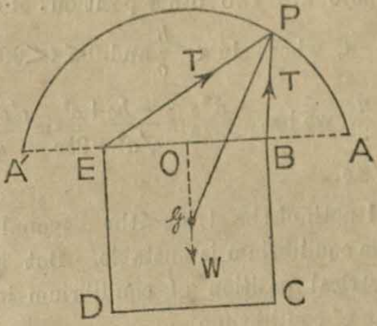
When $\theta=0, \frac{d^2z}{d\theta^2}=-\frac{c^2}{a}<0$ $\therefore$ z is a max. $\therefore$ Equilibrium is stable.

When $\theta=\frac{\pi}{2}, \frac{d^2z}{d\theta^2}=\frac{c^2}{\sqrt{a^2-c^2}}>0$ $\therefore$ z is a min. $\therefore$ Equilibrium is unstable.

Hence the horizontal position of equilibrium is unstable, but the vertical position is stable.

Ex. 15. A square board is hung flat against a wall by means of a string fastened to the two extremities of the upper edge and hung round a perfectly smooth rail; when the length of the string is less than the diagonal of the board, show that there are three positions of equilibrium. Show that the position of symmetry is unstable. [C. H. 1961]

Let $EBCD$ be the square and O the middle point of the side EB, the string being attached at E and B. Let P be the rail then since $EP+PB=$length of the string$=2a$, say, a constant, $\therefore$ P lies on the ellipse whose foci are E and B and whose major axis is of length $2a$. Since the square is in equilibrium under the action of the two tensions T, T of the strings EP, BP



and the vertical weight W of the square at its c. g. G, it follows that in positions of equilibrium PG is vertical. Let $EB=2c$, then

$$b^2=a^2-a^2e^2=a^2-EO^2=a^2-c^2$$

Take O as origin and OA as x-axis and let P be $(a\cos\theta, b\sin\theta)$. The c. g. of the square G is $(0, -c)$.

$\therefore$ z denoting the depth of G below P *i.e.* PG we have,

$$z=\sqrt{a^2\cos^2\theta+(b\sin\theta+c)^2}$$

$$=\sqrt{a^2-c^2\sin^2\theta+2bc\sin\theta+c^2}.$$

$$\therefore\ \frac{dz}{d\theta}=\frac{bc\cos\theta-c^2\sin\theta\cos\theta}{\sqrt{a^2+c^2+2bc\sin\theta-c^2\sin^2\theta}}$$

$$\frac{d^2z}{d\theta^2}=\frac{-bc\sin\theta-c^2(\cos^2\theta-\sin^2\theta)}{\sqrt{a^2+c^2+2bc\sin\theta-c^2\sin^2\theta}}$$

$$-\frac{(bc\cos\theta-c^2\sin\theta\cos\theta)^2}{(a^2+c^2+2bc\sin\theta-c^2\sin^2\theta)^{\frac{3}{2}}}.$$

For positions of equilibrium, $\frac{dz}{d\theta}=0$

$\therefore$ $c\cos\theta(b-c\sin\theta)=0$; $\cos\theta=0$ gives $\theta=\frac{\pi}{2}$.

$\therefore$ rail P is at $(0, b)$.

This gives the symmetrical position of equilibrium in which EB is horizontal.

$b-c\sin\theta=0$ gives $\sin\theta=\frac{b}{c}=\frac{\sqrt{a^2-c^2}}{c}$;

θ is real if $a^2-c^2<c^2$ or, $a^2<2c^2$ or, $2a<2\sqrt{2}.c$.

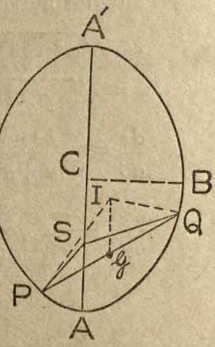
$\therefore$ If the length of the string is less than the diagonal of the board, then there are two other positions of equilibrium given by $\theta=\alpha$ and $\pi-\alpha$, where $\sin\alpha=\frac{b}{c}$ and $0<\alpha<90°$.

When $\theta=\frac{\pi}{2}$, we have $\frac{d^2z}{d\theta^2}=\frac{-bc+c^2}{\sqrt{a^2+2bc}}=\frac{c(c-\sqrt{a^2-c^2})}{\sqrt{a^2+2bc}}\gtreqless 0$ acc. as $2a\lesseqgtr 2\sqrt{2}.c$.

Thus if the length of the string < the diagonal of the board, z is a min. and the equilibrium is unstable. But in the contrary case, the symmetrical position of equilibrium is stable and it is the only position of equilibrium.

Ex. 16. A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse; if the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium it will pass through the focus.

Let PQ be the beam and G its mid-point. Let the normals at P and Q meet in I. Then the rod will be in equilibrium only if IG is vertical i.e. the ordinates of I and G be equal. Let P, Q be $(a\cos\theta,\ b\sin\theta)$, $(a\cos\phi,\ b\sin\phi)$ and let I be $(\xi,\ \eta)$ and G be $(\bar{x},\ \bar{y})$.



Then $2\bar{y}=b(\sin\theta+\sin\phi)$

$$=2b\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

and $\eta=\frac{b^2-a^2}{b}\sin\theta\sin\phi.\frac{\sin\frac{\theta+\varphi}{2}}{\cos\frac{\theta-\varphi}{2}}$.

$\therefore$ $\bar{y}=\eta$ if and only if

$$b\sin\frac{\theta+\varphi}{2}\cdot\cos\frac{\theta-\varphi}{2}-\frac{b^2-a^2}{b}\sin\theta\sin\phi\frac{\sin\frac{\theta+\varphi}{2}}{\cos\frac{\theta-\varphi}{2}}=0$$

$$or,\quad \sin\frac{\theta+\varphi}{2}\left[b^2\cos^2\frac{\theta-\varphi}{2}-(b^2-a^2)\left(\sin^2\frac{\theta+\varphi}{2}-\sin^2\frac{\theta-\varphi}{2}\right)\right]=0$$

$$or,\quad \sin\frac{\theta+\varphi}{2}=0 \quad or,\quad a^2\cos^2\frac{\theta-\varphi}{2}-a^2e^2\cos^2\frac{\theta+\varphi}{2}=0$$

$$or,\quad \sin\frac{\theta+\varphi}{2}=0 \quad or,\quad \cos\frac{\theta-\psi}{2}=\pm e\cos\frac{\theta+\varphi}{2},$$

Equation of PQ is $\frac{x}{a}\cos\frac{\theta+\varphi}{2}+\frac{y}{b}\sin\frac{\theta+\varphi}{2}=\cos\frac{\theta-\varphi}{2}$.

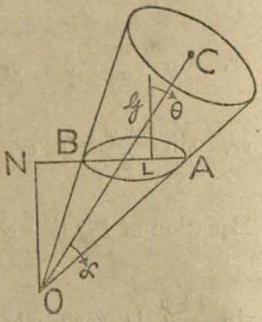
$\sin\frac{\theta+\varphi}{2}$ is zero when PQ is ∥ to y-axis i.e. horizontal and $\cos\frac{\theta-\varphi}{2}=\pm e\cos\frac{\theta+\varphi}{2}$ when PQ passes through either focus $(\pm ae, 0)$. Thus when the beam is in equilibrium it must be either horizontal or it must pass through one of the two focii. When it passes through a focus the stability of equilibrium can be easily determined. Let it pass through the lower focus S and let PL, QM, GN be drawn $\perp^s$ on the lower directrix. Then $SP=e.PL$ and $SQ=e.QM$.

$\therefore\ GN=\frac{1}{2}(PL+QM)=\frac{1}{2e}(SP+SQ)\geqslant\frac{1}{2e}\cdot PQ$, a constant [from $\triangle SPQ$].

$\therefore$ GN is least when PSQ is one straight line. Thus when PQ passes through S, the height of G above the lower directrix is a minimum and $\therefore$ the equilibrium is stable. Similarly when PQ passes through the upper focus, its distance from the upper directrix i.e. its depth below the upper directrix is least $\therefore$ the equilibrium is unstable. In order that PQ may pass through a focus its length must not be less than the latus rectum of the ellipse.

Ex. 17. A uniform solid right circular conical plug, whose height is h and semi-vertical angle α, is placed vertex downwards in a smooth circular hole of radius a, cut in a horizontal table. If $16a>3h\sin 2\alpha$, apply the energy test to show that there are three positions of equilibrium of the cone, and that the position with axis vertical is the stable one. [C. H. 1962]

Let the conical plug of vertex O, touch the circular hole at the points A and B. Then AB is a diameter of the hole and it is also the major axis of the ellipse in which the plane of the hole cuts the cone. Let the axis of the cone be inclined at an angle θ to the vertical, and let G be its c. g. $\therefore\ OG=\frac{3}{4}h$. From O draw $ON \perp$ to the plane of the hole. Then N is on AB and $\angle NOG=\theta$, $\angle NOB=\theta-\alpha$, $\angle NOA=\theta+\alpha$. If z=height of G above the plane of the hole, we have



$$z=GL=OG\cos\theta-ON=\tfrac{3}{4}h\cos\theta-ON.$$

$$\text{Now } AB=NA-NB=ON[\tan(\theta+\alpha)-\tan(\theta-\alpha)]$$

$$=\frac{ON.\sin 2\alpha}{\cos(\theta+\alpha)\cos(\theta-\alpha)}$$

$$\therefore\quad z=\tfrac{3}{4}h\cos\theta-\frac{2a\cos(\theta+\alpha)\cos(\theta-\alpha)}{\sin 2\alpha}$$

$$=\tfrac{3}{4}h\cos\theta-\frac{a}{\sin 2\alpha}(\cos 2\theta+\cos 2\alpha).$$

$$\therefore\quad \frac{dz}{d\theta}=-\tfrac{3}{4}h\sin\theta+\frac{2a}{\sin 2\alpha}.\sin 2\theta$$

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h\cos\theta+\frac{4a}{\sin 2\alpha}.\cos 2\theta.$$

For positions of equilibrium, we have $\frac{dz}{d\theta}=0$,

$$\therefore\quad \sin\theta\left[-\frac{3}{4}h+\frac{4a}{\sin 2\alpha}\cdot\cos\theta\right]=0$$

$$\therefore\quad \theta=0,\quad or,\quad \cos\theta=\frac{3h\sin 2\alpha}{16a}.$$

$\theta=0$ gives the vertical position of equilibrium in which the axis of the cone is vertical.

If $16a>3h\sin 2\alpha$, then θ is real and there are other positions of equilibrium; if $\frac{3h\sin 2\alpha}{16a}=\cos\beta$ where $0<\beta<90°$, then $\cos\theta=\cos\beta$ and $\therefore\ \theta=\beta$ or $-\beta$. Thus there are two other positions of equilibrium in which the axis of the cone makes the same angle with the vertical but on the opposite sides.

When $\theta=0$, $\dfrac{d^2z}{d\theta^2}=-\frac{3}{4}h+\dfrac{4a}{\sin 2\alpha}=\dfrac{1}{4\sin 2\alpha}[16a-3h\sin 2\alpha]>0$

$\therefore$ z is a minimum and the equilibrium is stable;

when $\theta=\pm\beta$, $\dfrac{d^2z}{d\theta^2}=-\frac{3}{4}h\cos\beta+\dfrac{4a}{\sin 2\alpha}\cos 2\beta$

$$=-\tfrac{3}{4}h.\frac{3h\sin 2\alpha}{16a}+\frac{4a}{\sin 2\alpha}\left[\frac{2\times 9h^2\sin^2 2\alpha}{16^2.a^2}-1\right]$$

$$=-\frac{9h^2\sin 2\alpha}{64a}[1-2]-\frac{4a}{\sin 2\alpha}$$

$$=\frac{9h^2\sin^2 2\alpha-16^2a^2}{64a\sin 2\alpha}<0$$

$\therefore$ z is a max. and the equilibrium is unstable.

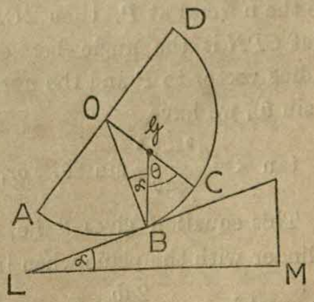
If $16a<3h\sin 2\alpha$, then the vertical position is the only equilibrium position. In this case, when $\theta=0$,

$$\frac{d^2z}{d\theta^2}=\frac{1}{4\sin 2\alpha}[16a-3h\sin 2\alpha]<0$$

$\therefore$ z is a max. and the equilibrium is unstable.

Hence when $16a>3h\sin 2\alpha$, there are three positions of equilibrium, the vertical one is stable and the inclined ones are unstable and when $16a<3h\sin 2\alpha$, the vertical position is the only position of equilibrium and it is unstable.

Ex. 18. A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha<\sin^{-1}\frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.



Let $ABCD$ be the solid hemisphere which rests in equilibrium on the plane touching the latter at B, and let its plane face AOD make an angle θ with the horizontal. Let OC be the radius $\perp$ to face AOD. Then the c. g. of the hemisphere, G, is on OC and $OG=\frac{3}{8}a$, where a is the radius. For equilibrium, the vertical through B must pass through G. $\therefore$ BG is vertical and $\angle OBG=\alpha$. From triangle OBG, we have,

$$\frac{OB}{\sin\theta}=\frac{OG}{\sin\alpha}\quad\therefore\quad\sin\theta=\frac{\sin\alpha}{8}$$

For θ to be real, we must have, $\sin\alpha < \frac{3}{8}$, which condition is given to be satisfied.

$$\therefore\quad \theta=\sin^{-1}\left(\frac{8\sin\alpha}{3}\right) \text{ gives the position of equilibrium.}$$

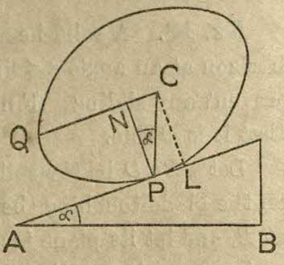
For stability, we have, $h=BG$ and α is the angle the normal BO at B makes with the vertical BG. Also $\rho_1=a$

$$\therefore\quad \text{Equilibrium is stable if } \frac{\cos\alpha}{h}>\frac{1}{a} \quad or,\quad h<a\cos\alpha.$$

$$\text{Now, } \frac{BG}{\sin(\theta-\alpha)}=\frac{OB}{\sin\theta} \quad \therefore\quad h=\frac{a\sin(\theta-\alpha)}{\sin\theta}$$
$$=a\cos\alpha-a\sin\alpha\cot\theta$$

As θ is acute, $h<a\cos\alpha$ and $\therefore$ the equilibrium is stable.

Ex. 19. A cylinder whose section is an ellipse of semi-axes a, b rests with its axis horizontal on a rough plane of inclination α; prove that if $\sin\alpha\leqslant\frac{a^2-b^2}{a^2+b^2}$, there are two positions of equilibrium, that one being stable in which the diameter of the cross-section which is parallel to the plane is greater than the vertical diameter.



The fig. shows the section of the cylinder by the plane $\perp$ to its axis and passing through its c. g. C. For equilibrium, the vertical through P, the point of contact, passes through C, the c g. of the cylinder. If PN be the normal at P, then $\angle CPN=\alpha$. But CPN is the angle between the radius vector to P and the normal at P. So if P be $(a\cos\theta,\ b\sin\theta)$, we have,

$$\tan\alpha=\frac{a^2-b^2}{2ab}\sin 2\theta. \quad or,\quad \sin 2\theta=\frac{2ab}{a^2-b^2}\tan\alpha.$$

This equation gives θ i.e. P, the point of contact of the cylinder with the plane when in equilibrium. For θ to be real we must have $\frac{2ab}{a^2-b^2}\tan\alpha\leqslant 1$

$$or,\quad 4a^2b^2\sin^2\alpha\leqslant(a^2-b^2)^2(1-\sin^2\alpha)$$

$$or,\quad (a^2+b^2)^2\sin^2\alpha\leqslant(a^2-b^2)^2$$

or, $\sin \alpha \leqslant \dfrac{a^2-b^2}{a^2+b^2}$

Thus when this condition is satisfied, θ has real values given by $\sin 2\theta = \sin 2\beta$, where $\sin 2\beta = \dfrac{2ab}{a^2-b^2} \tan \alpha$ and $0 < 2\beta < 90°$

$$\therefore \quad 2\theta = 2\beta \quad or, \quad \pi - 2\beta.$$

$$\text{i.e.,} \quad \theta = \beta \quad or, \quad \frac{\pi}{2} - \beta.$$

Hence there are two positions of equilibrium.

Draw CL perpendicular from C on the tangent AP at P.

Let $CL = p$, $CP = h$. Then $p = h \cos \alpha$. The radius of curvature of the ellipse at P is $\rho = \dfrac{a^2b^2}{p^3}$. Also, we know, $p.CQ = ab$, where CP, CQ are conjugate diameters and so CQ is parallel to AP, the tangent at P.

The equilibrium is stable or unstable, according as,

$$\frac{\cos \alpha}{h} \gtrless \frac{1}{\rho} \quad or, \quad \frac{a^2b^2}{p^3} . \frac{p}{h} \gtrless h$$

$$or, \quad \frac{a^2b^2}{a^2b^2} . CQ^2 \gtrless h^2 \quad \text{i.e.,} \quad CQ \gtrless h. \quad or, \quad 2CQ \gtrless 2h$$

$2CQ$ is the diameter parallel to the plane and $2h$ is the length of the vertical diameter.

Hence the equilibrium is stable or unstable according as the diameter parallel to the plane is greater or less than the vertical diameter.

Ex. 20. An elliptic disc of semi-axes a and b, slides in a vertical plane so as always to be in contact with two smooth rods, OL and OM which are in the same vertical plane and at right angles. Show that in the stable positions of equilibrium the major axis of the ellipse is parallel to one or other of the rods, whilst in the unstable position it is inclined at an angle θ to OL given by $\sin^2 \theta = \dfrac{a^2 \sin^2 \alpha - b^2 \cos^2 \alpha}{a^2 - b^2}$, where α is the inclination of OL to the vertical.

Let the disc in the position of equilibrium touch the rods OL, OM at Q and P and let the normals at Q and P meet in I. Then

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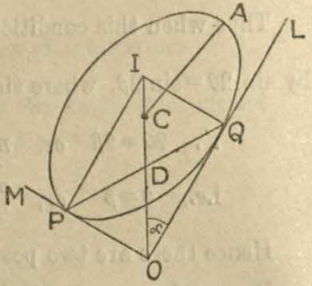
CI is vertical where C is the centre of the disc, for the forces which keep the disc in equilibrium are the two normal reactions along PI and QI and the weight of the disc which is vertical and acts at C. Now $OPIQ$ is a rectangle. $\therefore$ OI, PQ bisect each other at D. $\therefore$ I lies on OD. Again, CD is the diameter which bisects chords parallel to PQ. $\therefore$ CD passes through O, the point of intersection of the tangents at P and Q. $\therefore$ C lies on OD. Hence $ODCI$ is one straight line and it is vertical. $\therefore$ $\angle COQ=\alpha$. Now the perp. CN from C on OQ is $OC \sin\alpha$ and it makes with axis CA an angle $(90^\circ-\theta)$. $\therefore$ Equation of OQ is $x\sin\theta+y\cos\theta= OC\sin\alpha$. As OQ touches the ellipse, the condition of tangency gives $OC^2\sin^2\alpha=a^2\sin^2\theta+b^2\cos^2\theta$. Now as O is on the director circle, $OC^2=a^2+b^2$



$$\therefore\quad (a^2+b^2)\sin^2\alpha=a^2\sin^2\theta+b^2(1-\sin^2\theta)$$

$$or,\quad \sin^2\theta=\frac{a^2\sin^2\alpha-b^2\cos^2\alpha}{a^2-b^2}.$$

In the foregoing discussion we have assumed I to be different from C. But if they coincide, then CP, CQ become perpendicular conjugate diameters i.e , the axes and OL, OM are then parallel to the axes. Hence in this case, the major axis is parallel to OL or OM. For stability it is to be noted, that in all positions of the disc, its centre C is at a constant distance $\sqrt{a^2+b^2}$ from the fixed point O. Therefore the c. g., C, of the disc describes an arc of a circle whose centre is O. When OC is vertical, C is at the highest point of the arc and its height above O is a max. and $\therefore$ the equilibrium is unstable. When the major axis is parallal to OL or OM, C is at an end of the arc and its height above O is a min. and $\therefore$ the equilibriums are stable. Hence the results follow.

Analytical solution :

If p_1, p_2 be the perps. from C on the rods OL, OM we have $p_1=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ and $p_2=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$, where θ

is the inclination of the major-axis to OL. If z be the height of the c. g. C above O, we have z=proj. of OC on vertical through O.

$$\therefore \quad z=p_1 \sin \alpha+p_2 \cos \alpha$$

$$=\sin \alpha\sqrt{a^2 \sin^2\theta+b^2 \cos^2\theta}+\cos \alpha\sqrt{a^2 \cos^2\theta+b^2 \sin^2\theta}.$$

$$\therefore \quad \frac{dz}{d\theta}=(a^2-b^2) \sin \theta \cos \theta.\left[\frac{\sin \alpha}{\sqrt{a^2 \sin^2\theta+b^2 \cos^2\theta}}\right.$$

$$\left.-\frac{\cos \alpha}{\sqrt{a^2 \cos^2\theta+b^2 \sin^2\theta}}\right]$$

$$=(a^2-b^2) \sin \theta \cos \theta\left[\frac{\sin \alpha}{p_1}-\frac{\cos \alpha}{p_2}\right].$$

$$\frac{d^2z}{d\theta^2}=(a^2-b^2)\left[\cos 2\theta \left(\frac{\sin \alpha}{p_1}-\frac{\cos \alpha}{p_2}\right)\right.$$

$$\left.+\sin \theta \cos \theta(a^2-b^2) \sin \theta \cos \theta\times\left\{\frac{-\sin \alpha}{p_1^{\frac{3}{2}}}+\frac{-\cos \alpha}{p_2^{\frac{3}{2}}}\right\}\right].$$

For equilibrium,

$$\sin \theta \,.\, \cos \theta\left[\frac{\sin \alpha}{\sqrt{a^2 \sin^2\theta+b^2 \cos^2\theta}}-\frac{\cos \alpha}{\sqrt{a^2 \cos^2\theta+b^2 \sin^2\theta}}\right]=0$$

$$\therefore \quad \theta=0, \quad or, \quad \theta=\frac{\pi}{2},$$

$$or, \quad (a^2 \cos^2\theta+b^2 \sin^2\theta) \sin^2\alpha=(a^2 \sin^2\theta+b^2 \cos^2\theta) \cos^2\alpha$$

$$or, \quad \tan^2\theta=\frac{a^2 \sin^2\alpha-b^2 \cos^2\alpha}{a^2 \cos^2\alpha-b^2 \sin^2\alpha}=\frac{a^2 \tan^2\alpha-b^2}{a^2-b^2 \tan^2\alpha}\cdots\cdots(1)$$

We shall assume $\alpha>45^\circ$.

When $\theta=0$,

$$\frac{d^2z}{d\theta^2}=(a^2-b^2).\left(\frac{\sin \alpha}{b}-\frac{\cos \alpha}{a}\right)=(a^2-b^2).\frac{\cos \alpha}{a}\left[\frac{a}{b} \tan \alpha-1\right]>0$$

$\therefore$ z is a min. and the equilibrium is stable.

When $\theta=\frac{\pi}{2}$, $\frac{d^2z}{d\theta^2}=(a^2-b^2).\left[-\left(\frac{\sin \alpha}{a}-\frac{\cos \alpha}{b}\right)\right]$

$$=(a^2-b^2).\frac{\cos \alpha}{b}\left[1-\frac{b}{a} \tan \alpha\right]\gtrless 0 \text{ acc. as}$$

$$\tan \alpha\lessgtr\frac{a}{b}$$

$\therefore$ z is a min. or a max. and the equilibrium is stable or unstable acc. as $\tan\alpha \lessgtr \frac{a}{b}$. As $\alpha > 45^\circ$, $\tan\alpha > 1$ and $a^2 > b^2$

$\therefore$ in (1), $\tan\theta$ will be real if $a^2 - b^2\tan^2\alpha > 0$ *or*, $\tan\alpha < \frac{a}{b}$.

So if $\tan\alpha < \frac{a}{b}$, there is another position of equilibrium given by (1) and when θ has the value given by (1), we have

$$\frac{d^2z}{d\theta^2} = (a^2 - b^2)^2 \sin^2\theta \cos^2\theta \text{ [a negative number]} < 0$$

$\therefore$ z is a max and the equilibrium is unstable.

Thus if $\tan\alpha < \frac{a}{b}$, there are two positions of equilibrium in which the major-axis is parallel to either of the rods and these are stable and there is another position of equilibrium which is unstable ; and if $\tan\alpha > \frac{a}{b}$, there are only two positions of equilibrium, the first two, and the position in which the major-axis is ∥ to OL is stable and position in which it is ⊥ to OL is unstable. If ϕ be the inclination of OC to OL we have $\tan\phi = \frac{p_1}{p_2}$ $= \tan\alpha$, when θ is given by (1) $\therefore$ $\phi = \alpha$ and $\therefore$ OC is vertical in this case.

EXAMPLES A

1. A uniform beam, of thickness $2b$, rests symmetrically on a perfectly rough horizontal cylinder of radius a ; Show that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.

2. A solid circular cylinder of radius a and height h has one end in the shape of a hemisphere ; find the condition that it will be in stable equilibrium when standing on that end on a smooth horizontal plane, with its axis vertical.

3. A solid hemisphere rests in equilibrium on a fixed sphere of double its radius ; find whether the equilibrium is stable or not when the hemisphere rests on its curved surface or its plane face.

4. A uniform hemisphere rests in equilibrium with its curved surface on top of a sphere of double its radius. Show that the greatest weight that can be placed at the centre of the plane face without making the equilibrium unstable is one-eighth of the weight of the hemisphere.

5. A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

6. A sphere of weight W and radius r lies within a fixed spherical shell of radius R, and a particle of weight w is attached to its highest point. Show that the equilibrium is stable if $W > \frac{R-2r}{r} \cdot w$

7. A uniform lamina in the form of an equilateral triangle ABC of side $\frac{4a}{\sqrt{3}}$ rests in equilibrium with BC on the top of a fixed sphere of radius a. What is the greatest weight which can be attached to the vertex A so that the equilibrium may remain stable ?

8. A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. Show that the least radius of the sphere for which the equilibrium will be stable is $\frac{a}{2}$.

9. A heavy uniform hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $\frac{R}{r} > \sqrt{5} - 1$, the equilibrium is stable, whatever be the weight of the particle.

10. A body consisting of a plane lamina in the form of a rectangle $ABCD$ and a semi-circle AED attached to the side AD, where E is the middle point of the arc AED, is balanced vertically with the point E in contact with a horizontal plane. Show that the equilibrium is unstable if the ratio $AB : AD$ is greater than $\frac{1}{\sqrt{6}}$.

11. A heavy uniform cube balances on the highest point of a sphere, whose radius is r. If the sphere be rough enough to prevent sliding, and if the side of the cube be $\frac{\pi r}{2}$, show that the cube can rock through a right angle without falling.

12. A solid ellipsoid, whose axes are of lengths $2a$, $2b$, $2c$ rests with the "c-axis" vertical on a rough horizontal plane. The centre of gravity is on the vertical axis at a distance h from the bottom vertex. Show that the equilibrium is stable if h is less than both $\frac{a^2}{c}$ and $\frac{b^2}{c}$.

13. A solid of uniform density in the form of a portion of a paraboloid of revolution, bounded by a plane perpendicular to the axis is placed with its axis vertical and vertex resting on a horizontal plane. If the paraboloid is formed by the revolution of a parabola of latus rectum $4a$, show that the equilibrium will be stable if the height of the paraboloid is less than $3a$.

[C. H. 1966]

14. A lamina in the form of a cycloid, whose generating circle is of radius a, rests on the top of another cycloid whose generating circle is of radius b, their vertices being in contact and their axes vertical. If h be the height of the c. g. of the upper cycloid above its vertex, show that the equilibrium is stable if $h < \frac{4ab}{a+b}$.

15. A stiff wire in the form of a parabola rests on the ground with its plane vertical. The c. g. of the wire is on the axis of the parabola at a distance h from the vertex, and the latus rectum is $4a$. Prove that, if $h > 2a$, there is a position of equilibrium in which the axis makes an angle $\tan^{-1}\left(\frac{a}{h-2a}\right)^{\frac{1}{2}}$ with the horizon and that this position of equilibrium is stable.

16. A uniform plank of thickness $2h$ rests across the top of a fixed circular cylinder of radius a, whose axis is horizontal. Prove that the gain of potential energy when the plank is turned, without slipping, through an angle θ in a vertical plane parallel to its length is

$$W\{a\theta \sin \theta - (a+h)(1-\cos \theta)\},$$

and deduce the condition of stability.

17. Two equal particles repel each other according to the fifth power of the distance, and are connected by an elastic string. Find the position of equilibrium, and show that it is stable if the extension of the string is less than one-quarter of its original length.

EXAMPLES B

1. A uniform heavy rod AB rests against a smooth vertical wall and over a smooth peg C. Find the position of equilibrium, and determine whether it is stable or unstable.

2. A, B are two points on the same level at a distance a. A uniform rod AC of length a and weight $2\sqrt{3}$ lb is hinged at A and is supported by a string fastened at C, passing over a smooth pulley at B, and carrying a weight of one pound. Find the equilibrium position, and show that it is stable. [C. H. 1959]

3. A light string is attached to a fixed point O and passes over a fixed smooth pulley P, so that $OP = 2a$ is horizontal, and supports a weight W_1 attached to its end. A second weight $W_2 < 2W_1$ slides freely on the string between O and P. Show that there is one position of equilibrium and that it is stable. [C. H. 1962]

4. A weight W is supported on a smooth inclined plane by a given weight P, connected with W by means of a string passing round a fixed pulley whose position is given. Find the position of equilibrium of W on the plane, and show that it is stable.

5. A uniform square board is capable of motion in a vertical plane about a hinge at one of its angular points ; a string attached to one of the nearest angular points, and passing over a pulley vertically above the hinge at a distance from it equal to a side of the square supports a weight whose ratio to the weight of the board is $1 : \sqrt{2}$. Find the positions of equilibrium, and determine whether they are respectively stable or unstable. [M. T.]

6. A rhombus, formed of four equal rods freely jointed, is placed symmetrically over a fixed smooth vertical circle, so that only the upper pair are in contact with the circle. If $\sphericalangle$ be the angle a rod makes with the vertical in the position of equilibrium, $2a$ the length of each rod and c the radius of the circle, show that $2a \sin^3\sphericalangle = c \cos \sphericalangle$; and that the equilibrium is stable. [C. H. 1960]

7. A uniform rectangular lamina $ABCD$ rests in a vertical plane with A in contact with a smooth vertical wall and B attached by an inextensible string to a point E of the wall vertically above A. $AB = BE = 2x$, and $AD = 2y$. If the inclination of the string to the wall is θ, find the depth of the c. g. below E in terms of θ. Find the position of equilibrium and determine whether it is stable or unstable. [C. H. 1965]

8. AB is a uniform rod, of length $6a$ and weight W, which can turn freely about a fixed point in its length distant $2a$ from A. AC and BC are light strings of length $5a$ attached to a particle C of weight ω. Show that if W is less than 2ω there will be stable equilibrium with AB inclined to the horizontal at an angle $\tan^{-1}\left(\frac{W+\omega}{4\omega}\right)$.

9. A uniform square board of mass M is supported in a verticle plane on two smooth pegs at the same horizontal level. The distance between the pegs is a and the diagonal of the square is $D > 4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable if $4am > M(D-4a)$.

10. A uniform elliptic plate of semi-axes a, b rests in a vertical plane on two smooth pegs P, Q at the same level. Prove that unsymmetrical positions of equilibrium are possible if PQ lies between $a\sqrt{2}$ and $b\sqrt{2}$. Examine the stability of various positions of equilibrium.

11. An isosceles triangular lamina of angle $2\sphericalangle$ rests between two smooth pegs at the same level, distant $2c$ apart ; prove that if h be the distance of the centre of gravity from the vertex, and if $2c \sec \sphericalangle < h < \frac{2c}{\sin \sphericalangle \cos \sphericalangle}$, then oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position in case $h \geqslant \frac{2c}{\sin \sphericalangle \cos \sphericalangle}$.

12. ABC is a triangular lamina, right angled at A, and its c. g. G is in the bisector of the angle BAC at a distance kl from A. If rests in a vertical plane with AB and AC in contact with two smooth pegs in the same

horizontal line at a distance l apart, and with A lower than the pegs. When AG makes an angle θ with the vertical, show that the vertical height of G above the level of the pegs is $l\,(k \cos \theta - \frac{1}{2} \cos 2\theta)$. Examine the stability of equilibrium in the various possible cases. [M. T.]

13. A rectangular picture-frame hangs from a small perfectly smooth pulley by a string of length $2a$ attached symmetrically to two points on the upper edge at a distance $2c$ apart. Prove that if the depth of the picture is less than $\dfrac{2c^2}{\sqrt{a^2 - c^2}}$, there are three positions of equilibrium, of which the symmetrical one is unstable. If the depth exceed the above value the symmetrical position of equilibrium is the only one and is stable.

14. An elliptic cylinder is placed with its axis horizontal on a rough plane inclined to the horizon at an angle less than the angle of friction; prove that the cylinder cannot rest if the inclination of the plane exceeds $\sin^{-1}\left(\dfrac{a^2 - b^2}{a^2 + b^2}\right)$; and if the inclination is equal to $\sin^{-1}\left(\dfrac{a^2 - b^2}{a^2 + b^2}\right)$, the equilibrium is neutral to a first approximation.

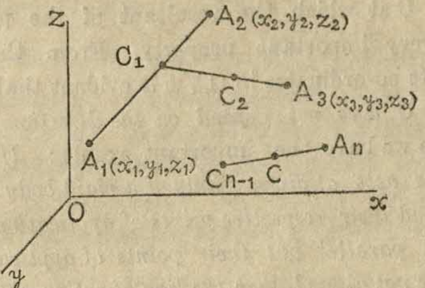
15. A uniform hemispherical shell rests upon an inclined plane. Show that for equilibrium, the inclination of the plane to the horizon must not exceed 30° and that the equilibrium is stable.

CHAPTER V

CENTRE OF GRAVITY

5·1. The centre of parallel forces.

Let any number of parallel forces P_1, P_2, P_3, ... act at the points A_1, A_2, A_3, ... of a rigid body. Let Ox, Oy, Oz be any cartesian system of axes, rectangular or oblique but fixed in the body, and let (x_1, y_1, z_1), (x_2, y_2, z_2), ..., be the co-ordinates of A_1, A_2, ... The forces P_1 acting at A_1 and P_2 acting at A_2 are equivalent to a single parallel force (P_1+P_2) acting at a point C_1 on A_1A_2 such that $P_1.A_1C_1=P_2.C_1A_2$.



Let (ξ_1, η_1, ζ_1) be the co-ordinates of C_1.

Since $\dfrac{A_1C_1}{C_1A_2}=\dfrac{P_2}{P_1}$, C_1 divides A_1A_2 in the ratio $P_2 : P_1$.

$\therefore$ From solid geometry we know,

$$\xi_1=\frac{P_1x_1+P_2x_2}{P_1+P_2},\ \eta_1=\frac{P_1y_1+P_2y_2}{P_1+P_2},\ \zeta_1=\frac{P_1z_1+P_2z_2}{P_1+P_2}.$$

The force (P_1+P_2) acting at C_1 and a third force P_3 acting at A_3 are again equivalent to a single parallel force $(P_1+P_2+P_3)$ acting at a point C_2 on C_1A_3 such that $(P_1+P_2)C_1C_2=P_3.C_2A_3$ and if (ξ_2, η_2, ζ_2) be the co-ordinates of C_2, we have in the same way, $\xi_2=\dfrac{(P_1+P_2)\xi_1+P_3x_3}{(P_1+P_2)+P_3}=\dfrac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}$, with similar expressions for η_2 and ζ_2.

Proceeding in this way till all the given forces are used up, we see that the given forces are equivalent to a single force $(P_1+P_2+P_3+\cdots)$, called their resultant, parallel to the given forces and acting at a point C, whose co-ordinates (ξ, η, ζ) are given by,

$$\left.\begin{aligned}\xi&=\frac{P_1x_1+P_2x_2+P_3x_3+\cdots}{P_1+P_2+P_3+\cdots}=\frac{\Sigma Px}{\Sigma P}\\ \eta&=\frac{P_1y_1+P_2y_2+P_3y_3+\cdots}{P_1+P_2+P_3+\cdots}=\frac{\Sigma Py}{\Sigma P}\\ \zeta&=\frac{P_1z_1+P_2z_2+P_3z_3+\cdots}{P_1+P_2+P_3+\cdots}=\frac{\Sigma Pz}{\Sigma P}\end{aligned}\right\}\cdots\cdots(1)$$

The point C at which the resultant of the parallel forces acts has a very important property. From the expressions obtained for its co-ordinates in (1), it is evident that the position of the point C *does not depend on the direction of the parallel forces.* Hence we have the important result : *If a system of parallel forces acts at given points of a rigid body and the forces are turned round their respective points of application so that the forces remain parallel but their points of application and their magnitudes are not altered, then the line of action of their resultant always passes through a point fixed in the body.* This point is called the '*centre of parallel forces.*'

It follows that if a system of parallel forces acts at different points of a rigid body and the body is moved in any manner in space but the directions of the forces and their points of application remain unchanged then the resultant of the system always passes through the same point in the body viz. the centre of the forces.

It is to be observed that the parallel forces P_1, P_2, ... are not necessarily all like. When unlike forces also occur, we have to impose the condition $\Sigma P \neq 0$.

5·2. The centre of gravity of a rigid body or a rigidly connected system of particles.

A body is made up of particles and each particle is attracted towards the Earth's centre with a vertical force called the weight of the particle. These weights may be supposed to from a system of parallel forces. They have a resultant, which is called the weight of the body. And, from the theory of parallel forces we find, there is a certain fixed point in the body through which the line of action of the weight always passes, however the body be moved in space. This point is called the *centre of gravity* of the body.

If $w_1, w_2, \ldots$ be the weights of the particles situated at the points (x_1, y_1, z_1), (x_2, y_2, z_2), ... referred to any cartesian system of axes, rectangular or oblique, then $(\bar{x}, \bar{y}, \bar{z})$ the co-ordinates of the centre of gravity of the body, are given by the formulae

$$\bar{x}=\frac{\Sigma wx}{\Sigma w},\ \bar{y}=\frac{\Sigma wy}{\Sigma w},\ \bar{z}=\frac{\Sigma wz}{\Sigma w}\ldots\ldots(1)$$

It is evident that a body can have only one centre of gravity.

If in the formulae (1), each w is replaced by kw, the values of $\bar{x}, \bar{y}, \bar{z}$ are not altered. Hence the centre of gravity of a body remains unaltered if the weights of the particles are altered in the same ratio. In particular, if the body is homogeneous the weights of the particles may be replaced by their volumes.

5·3. In finding the resultant of a system of parallel forces, the forces may be compounded in any order ; it follows therefore that if a system of parallel forces be divided into several groups $S_1, S_2, S_3, \ldots$ of which the resultants are $R_1, R_2, R_3, \ldots$ and the centres are $C_1, C_2, C_3, \ldots$ and if C be the centre of the forces $R_1, R_2, R_3, \ldots$ acting at the points $C_1, C_2, C_3, \ldots$ then C is the centre of the system. So in finding the centre of gravity of a body or a rigidly connected system of particles, we may divide the body or system into several parts, find their weights and the centres of gravity of these parts where their weights act and then find the centre of gravity of these weights. This final centre of gravity will be the centre of gravity of the body or system. Hence we have the important rule to find the c. g. of a body or a rigidly connected system of bodies : Divide the body or system into a number of suitable parts, finite in size or elementary *i.e.* infinitesimal, such that the weight and the c. g. of each part is known. If $w_1, w_2, \ldots$ be the wts. of these parts and (x_1, y_1, z_1), (x_2, y_2, z_2), ... be the co-ordinates of their c.g's, referred to axes, rectangular or oblique, then the co-ordinates $\bar{x}, \bar{y}, \bar{z}$ of the c. g. of the body or system are given by $\bar{x}=\frac{\Sigma wx}{\Sigma w}$, etc, or $\bar{x}=\frac{\int xdw}{\int dw}$, etc, according as the parts are finite in size or elementary.

Since weights are proportional to the masses and mass is volume × density, we can write the above formulae as

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}=\frac{\Sigma \rho vx}{\Sigma \rho v} \text{ etc.......}$$

or, $\bar{x}=\dfrac{\int xdm}{\int dm}=\dfrac{\int x\rho dv}{\int \rho dv}$, etc, m, v, ρ denoting mass, volume and density.

When the density is varying, what will be the most advantageous method of dividing the body or system, depends entirely on the law of variation of the density. Generally, the body or system should be so divided that each part is homogeneous so that its mass and c. g. are easily found. The law of symmetry is often helpful in finding the position of the c. g. If the body is homogeneous and symmetrical about a point, that point is its c. g. For examples, the middle point of a uniform rod is its c. g. ; the centre of a uniform circular ring or circular disc is the c. g. If the body be homogeneous and symmetrical about a line or about a plane, its c. g. lies on the line or on the plane.

In using the above rules it should be noticed that *some of the masses may be negative*, as in the theory of parallel forces, from which the formulae are deduced, some of the forces may have opposite signs. This fact is conveniently used in finding the c. g. of a body when a portion of it has been removed, and we know the masses and the c. g's of the whole body and the portion removed. Then the remaining portion whose c. g. is to be found is considered as composed of the two bodies : (1) the whole body, of positive mass and (2) the portion removed, of negative mass. For the c. g. of the remaining portion, is the point where its weight acts and its weight is the resultant of the weight of the whole body and the weight of the removed portion *reversed in direction*. For this reversal of direction, the mass of the removed portion is regarded as negative.

5·4. The centre of gravity of some elementary bodies :

We shall simply state the positions of the c. g's of some elementary bodies without going into the proof. For proofs reference may be made to "Statics" by Basu and Ghosh.

(*a*) Four uniform rods forming a parallelogram or a uniform lamina in the form of a parallelogram—In both cases, the c. g. is the point of intersection of the straight lines joining the mid-points of the opposite sides. This point is also the point of intersection of the diagonals.

(*b*) A uniform triangular lamina—The c. g. is the point of intersection of the medians, which divides each median in the ratio 2 : 1. It is also the c. g. of three equal wts. placed at the vertices of the triangle. Hence in problems concerning the c. g. of bodies, *any uniform triangular lamina can be replaced by three equal wts. placed at the vertices of the triangle, each wt. being equal to $\frac{1}{3}$rd the wt. of the triangle.*

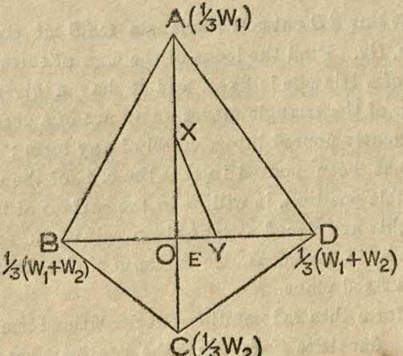
(*c*) Three uniform rods forming a triangle—The c. g. is the in-centre of the triangle formed by joining the middle-points of the rods. It is also the c. g. of wts. $(b+c)$, $(c+a)$, $(a+b)$ placed at the vertices A, B, C of the triangle ABC.

(*d*) Tetrahedron—The c. g. is on the line joining any vertex to the c. g. of the opposite face and one-fourth of the way up this line. It is also the c. g. of four equal wts. placed at the four vertices of the tetrahedron.

(*e*) Volume of a cone or Pyramid—The c. g. is on the line joining the vertex to the c. g. of the base and one-fourth of the way up the line.

(*f*) Surface of a right circular cone—The c. g. is on the line joining the vertex to the centre of the base and one-third of the way up the line.

5·5. To find the c. g. of a uniform quadrilateral lamina.



Let $ABCD$ be the quadrilateral lamina and let the diagonals AC, BD intersect at O. Divide the quadrilateral into the two triangles ABD and CBD and let their weights be W_1 and W_2 respectively so that $W=W_1+W_2$ is the weight of the quadrilateral. Since the triangles ABD, CBD are on the same base BD, therefore, we have, $\dfrac{W_1}{W_2}=\dfrac{AO}{CO}$ ... ... (1)

Replace each triangle by three equal weights at its vertices, each weight being equal to one-third the weight of the triangle. Then the c. g. of the quadrilateral is the same as the c. g. of the following weights :

(1) $\frac{1}{3}W_1$ at A, (2) $\frac{1}{3}W_2$ at C (3) $\frac{1}{3}(W_1+W_2)$ at each of B and D. Now $\frac{1}{3}W_1$ at A is equivalent to two weights, $\frac{1}{3}W$ at A and $(-\frac{1}{3}W_2)$ at A ; and $\frac{1}{3}W_2$ at C is equivalent to two weights, $\frac{1}{3}W$ at C and $(-\frac{1}{3}W_1)$ at C ; and finally, $(-\frac{1}{3}W_2)$ at A and $(-\frac{1}{3}W_1)$ at C are together equivalent to $(-\frac{1}{3}W)$ at O, since by (1) $W_1.CO=W_2.AO$. Thus the above weights are equivalent to a weight $\frac{1}{3}W$ at each of A, B, C and D and a negative weight $(-\frac{1}{3}W)$ at O. Hence the *c. g. of the quadrilateral lamina is the same as that of four equal weights, placed at the vertices of the quadrilateral together with an equal but negative weight placed at the intersection of the diagonals.*

If (x_1, y_1), (x_2, y_2), ……, (x_5, y_5) be the co-ordinates of the four vertices and the point of intersection of the diagonals, then the co-ordinates of the c. g., $(\bar{x}, \bar{y})$, are given by

$$\bar{x}=\tfrac{1}{3}(x_1+x_2+x_3+x_4-x_5), \quad \bar{y}=\tfrac{1}{3}(y_1+y_2+y_3+y_4-y_5).$$

EXAMPLES

1. A straight line AB cuts off a triangle OAB of constant area from two fixed lines Ox, Oy. Find the locus of the c. g. of the triangle OAB.

2. The base of a triangle is fixed, and it has a given vertical angle ; show that the c. g. of the triangle moves on an arc of a certain circle.

3. Water is gently poured into a vessel of any form ; prove that, when so much water has been poured in that the c. g. of the vessel and water is in the lowest possible position, it will be in the surface of the water. [M. T.]

4. Four weights are placed at four given points in space, the sum of two of the weights is given, as also the sum of the other two : prove that their c. g. lies on a fixed plane. [M. T.]

5. Three uniform thin rods of different densities form a triangle ABC. Prove that the densities are in the ratios $\sec A : \sec B : \sec C$ or $\operatorname{cosec}^2 \dfrac{A}{2} : \operatorname{cosec}^2 \dfrac{B}{2} : \operatorname{cosec}^2 \dfrac{C}{2}$ according as the c. g. of the rods is the circumcentre or the incentre of the triangle ABC. [S.]

6. If O be the intersection of the diagonals AC, BD of a quadrilateral $ABCD$ and points X and Y be taken on AC and BD respectively such that $AX = OC$ and $DY = OB$, show that the c. g. of the quadrilateral coincides with that of the triangle OXY.

7. If a quadrilateral $ABCD$ is divided into two equal triangles by the diagonal AC, prove that the c. g. of the quadrilateral is on AC and conversely.

8. If the c. g. of a quadrilateral be the same as that of four equal particles placed at its angular points, show that the c. g. is the point of intersection of the diagonals. Hence show that the quadrilateral must be a parallelogram.

9. From a triangular area ABC is cut off one nth. part of its area by means of a straight line parallel to BC; show that the c. g. of the remainder divides the median through A in the ratio $(n+\sqrt{n}-2) : 2(n+\sqrt{n}+1)$.

10. A square hole is punched out of a circular lamina, the diagonal of the square being a radius of the circle. Show that the c. g. of the remainder is at a distance $\frac{a}{8\pi - 4}$ from the centre of the circle whose diameter is a.

11. From a uniform triangular board a portion consisting of the area of the inscribed circle is removed; show that the distance of the c. g. of the remainder from the side a is $\frac{\Delta}{3as}\,\frac{2s^3 - 3\pi a\Delta}{s^2 - \pi\Delta}$, where Δ is the area and s the semi-perimeter of the board.

12. Prove that the centroid of the surface of a tetrahedron is the centre of the sphere inscribed in the tetrahedron formed by joining the centroids of the faces. [S]

13. A cone of semi-vertical angle $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is enclosed in the circumscribing sphere; show that it will rest in any position. [M. T.]

14. A solid right circular cone has its base scooped out, so that the hollow is a right cone on the same base; how much must be removed so that the c. g. of the remainder may coincide with the vertex of the hollow?

15. Into a thin cylindrical vase, of weight W and cross-section s, whose c. g. is distant b from its base is slowly poured liquid of density ρ. When the height of the c. g. of the whole is a minimum, show that the weight of the liquid poured in is $\sqrt{W(W+2sb\rho g)} - W$.

16. Two tangents are drawn to a circle of radius a subtending an angle 2α at the centre. Prove that the centre of gravity of the figure bounded by the tangents and the smaller arc between them is at a distance from the centre equal to $\frac{a}{3}\cdot\frac{\tan^2\alpha \sin\alpha}{(\tan\alpha - \alpha)}$. [S.]

5·6. Centre of gravity of a continuous distribution of matter.

To find the centre of gravity of a continuous distribution of matter forming a finite body of given shape, the summations in the general formulæ (1) of Art. 5·2 are to be replaced by definite integrals as already indicated in Art. 5·3.

Let a continuous quantity of matter be given. Let a suitable point O be taken as origin and Ox, Oy, Oz, three suitable lines as axes of co-ordinates, which may be rectangular or oblique. Let the given quantity of matter be divided, in any suitable manner, into elementary portions so that the mass and the centre of gravity of each portion is known; let dm be the mass of one such element and x, y, z the co-ordinates of its centre of gravity. Then $\bar{x}$, $\bar{y}$, $\bar{z}$, the co-ordinates of the centre of gravity of the given matter are given by the formulæ

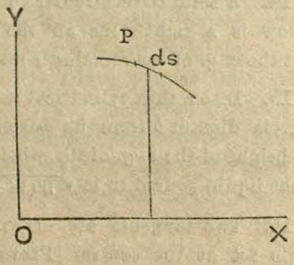
$$\bar{x}=\frac{\int x\,dm}{\int dm},\quad \bar{y}=\frac{\int y\,dm}{\int dm},\quad \bar{z}=\frac{\int z\,dm}{\int dm}.$$

The limits of integration are to be so chosen as to include the whole matter.

The given quantity of matter may be distributed in the form of (*a*) an arc of any curve, (*b*) a plane area of any shape, (*c*) a portion of a surface or (*d*) a portion of a solid. Moreover, the density may be uniform or variable.

5·7. Centre of gravity of any arc of a plane curve.

The given matter is arranged in the form of an arc of a curve whose equation is known. Let ds be the length of an elementary portion of the arc at P whose cartesion co-ordinates are x, y and whose arcual distance, measured from some fixed point of the curve, is s, and let ρ be the linear density at P. Then the mass of the element $dm=\rho ds$ and its centre of gravity is ultimately at P.



∴ co-ordinates of the centre of gravity of the element are x, y.

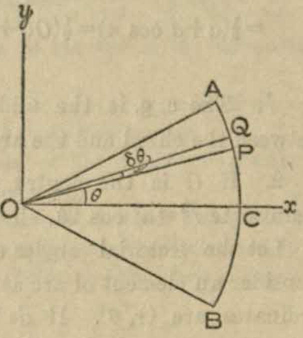
∴ the co-ordinates of the centre of gravity of the given arc are $\bar{x}=\frac{\int x.\rho ds}{\int \rho ds}$, $\bar{y}=\frac{\int y.\rho ds}{\int \rho ds}$, where $ds=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx$

or, $ds=\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta$ and $x=r\cos\theta$, $y=r\sin\theta$, according as the equation of the curve is given in the cartesian from $y=f(x)$ or in the polar form $r=f(\theta)$. The equation of the curve being known and ρ being given as a function of x, y or of r, θ the integrations can be performed, taking the proper limits.

Similar formulæ hold for a curve in three dimensions. We have $\bar{x}=\frac{\int x.\rho ds}{\int \rho ds}$, $\bar{y}=\frac{\int y.\rho ds}{\int \rho ds}$, $\bar{z}=\frac{\int z.\rho ds}{\int \rho ds}$. If the equation of the curve be given in the parametric form $x=x(t)$, $y=y(t)$, $z=z(t)$, then $ds=\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}}.dt.$

5·8. Centre of gravity of a uniform arc of a circle.

Let ACB be the given arc of a circle whose centre is O and radius is a. Let OC bisect the angle AOB and let angle $AOB=2\alpha$. We are taken O as origin and OC as axis of x. Let PQ be any element of the arc where P is (x, y), and angle POx is θ. Then ds, the length of the arc PQ, is $ad\theta$ and the abscissa of its c. g. is $x=a\cos\theta$. From symmetry, the c. g. of the arc is on the x-axis. If $\bar{x}$ be its distance from O, we have,



$$\bar{x}=\frac{\int_{-\alpha}^{\alpha}\rho.a\cos\theta.ad\theta}{\int_{-\alpha}^{\alpha}\rho\, ad\theta}=a.\frac{\sin\alpha}{\alpha}.$$

Hence the c. g. of the arc is on the bisector of its angle and at a distance from the centre equal to $\mathbf{a}\cdot\frac{\sin\alpha}{\alpha}$.

This result is of frequent use.

When the arc is a semi-circle, α *is* $\frac{\pi}{2}$ *and* $\bar{x}=\frac{2a}{\pi}$.

5·9. Examples :

1. If the density at any point of an arc of a uniform circular wire varies as its distance from the central radius, show that the c. g. is on the central radius and is midway between the chord and the arc.

We proceed as in Art 5·8. Here the density ρ is not constant but varies as $|y|$. Let $\rho=k|y|=k.a|\sin\theta|$, for element PQ. From symmetry the c. g. is on OC and

$$\bar{x}=\frac{\int_{-\alpha}^{\alpha} ka\,|\sin\theta|.\,a\cos\theta.\,ad\theta}{\int_{-\alpha}^{\alpha} ka\,|\sin\theta|.\,ad\theta}=\frac{2a\int_0^{\alpha}\sin\theta\cos\theta d\theta}{2\int_0^{\alpha}\sin\theta d\theta}$$

$$=a.\frac{1+\cos\alpha}{2}$$

$=\frac{1}{2}(a+a\cos\alpha)=\frac{1}{2}(OC+ON)$, if AN is $\perp$ on OC.
(not shown in fig).

$\therefore$ The c. g. is the middle point of NC, and so midway between the chord and the arc.

2. If G is the centre of gravity of any arc PQ of the lemniscate $r^2=a^2\cos 2\theta$, show that OG bisects the angle POQ.

Let the vectorial angles of P and Q be α and β respectively. Consider an element of arc at a point on the curve whose co-ordinates are (r,θ). If ds be its length and ρ the density, then mass of element is ρds and its c. g. is $(r\cos\theta, r\sin\theta)$.
$\therefore$ If $(\bar{x},\bar{y})$ be the co-ordinates of G, the c. g. of the arc PQ,

we have, $\bar{x}=\dfrac{\int\rho ds.\,r\cos\theta}{\int\rho ds.}$, $\bar{y}=\dfrac{\int\rho ds.\,r\sin\theta}{\int\rho ds.}$.

$\therefore\ \dfrac{\bar{y}}{\bar{x}}=\dfrac{\int r\sin\theta ds}{\int r\cos\theta ds}$. Now, differentiating equation of curve

we have, $r\dfrac{dr}{d\theta}=-a^2\sin 2\theta$. $\therefore\ \left(\dfrac{ds}{d\theta}\right)^2=\left(\dfrac{dr}{d\theta}\right)^2+r^2$

$$=a^2.\frac{\sin^2 2\theta}{\cos 2\theta}+a^2.\cos 2\theta=a^2\sec 2\theta.$$

$\therefore\ ds=a\sqrt{\sec 2\theta d\theta}.$

$$\therefore\quad \frac{\bar{y}}{\bar{x}}=\frac{\int_{\alpha}^{\beta} a.a.\ \sin\theta d\theta}{\int_{\alpha}^{\beta} a.a.\ \cos\theta d\theta}=\frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}=\tan\frac{\alpha+\beta}{2}.$$

$\therefore$ OG makes an angle $\frac{\alpha+\beta}{2}$ with the axis *i.e.* OG bisects the angle POQ.

EXAMPLES

1. Find the centroid of the arc of the parabola $y^2=4ax$ included between the vertex and an end point of the latus rectum.

2. Find the co-ordinates of the centre of gravity of the arc of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ which lies in the positive quadrant.

3. Show that the co-ordinates of the centre of gravity of the arc of the catenary $y=c\cosh\frac{x}{c}$ from $x=0$ to $x=x$ are $\bar{x}=x-\frac{c(y-c)}{s}$, $\bar{y}=\frac{1}{2}\left(y+\frac{cx}{s}\right)$.

4. Find the position of the centroid of the arc of the semi-cardioid $r=a(1+\cos\theta)$.

5. Find the equation of the line joining the centroid of the arc of half a loop of the lemniscate $r^2=a^2\cos 2\theta$ to the double point.

6. Find the centroid of the arc of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ which is in the positive quadrant.

7. If the density of a semi-circular arc varies as the square of the distance from the point which is the other end of the diameter through the middle point of the arc, show that the distance of the centre of gravity of the arc from the centre is $\frac{a}{2}\cdot\frac{\pi+4}{\pi+2}$.

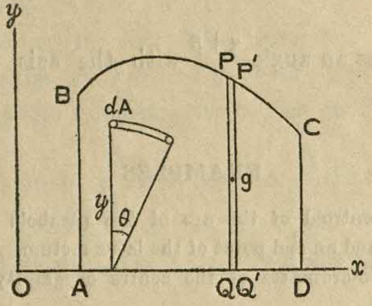
8. Find the centre of gravity of a uniform circular arc, the density varying as the length of the arc measured from one end.

9. Show that the centre of gravity of the arc of a cycloid in which the density at any point varies as the nth. power of its arcual distance from the vertex is on the axis at a distance 2 $\frac{n+1}{n+3}$ from the vertex.

10. Show that the centroid of the arc of the helix $x=a\cos\theta$, $y=a\sin\theta$, $z=b\theta$, lying between the points $\theta=0$ and $\theta=\alpha$ has the co-ordinates $\bar{x}=a.\ \frac{\sin\alpha}{\alpha}$, $\bar{y}=a.\ \frac{1-\cos\alpha}{\alpha}$, $z=\frac{1}{2}b\alpha$.

5·10. Centre of gravity of any plane area.

The given matter is distributed in the form of the area $ABCD$, bounded by the ordinates at A and D, the x-axis and the curve *BC* whose equation is known. We first assume the density ρ to be constant. Let the area be divided into thin elementary strips like $PP'Q'Q$ by lines drawn parallel to the y-axis. Let the co-ordinates of P be (x, y), then those of P' are $(x+dx, y+dy)$; let g be the centre of gravity of this elementary strip. The mass dm of the elementary strip may be taken to be ultimately equal to $\rho y dx$ and the co-ordinates of g may be taken to be ultimately equal to $x, \frac{y}{2}$. Therefore the co-ordinates of the centre of gravity of the given area *ABCD* are given by the formulæ $\bar{x}=\frac{\int x.\rho y dx}{\int \rho y dx}=\frac{\int xy dx}{\int y dx}$, $\bar{y}=\frac{\int \frac{1}{2}y.\rho y dx}{\int \rho y dx}=\frac{1}{2}.\frac{\int y^2 dx}{\int y dx}$; the limits of integration are $x=OA$ and $x=OD$.



If the given matter is in the form of a sectorial area *AOB* bounded by two radii vectors *OA*, *OB* and a curve *AB* whose polar equation is given (see fig Art. 5·8), we divide the area into elementary triangular strips like *OPQ*; let the polar co-ordinates of P be (r, θ) then those of Q are $(r+dr, \theta+d\theta)$. The mass dm of the elementary triangle *OPQ* may be taken to be ultimately equal to $\rho.\frac{1}{2}r^2 d\theta$ and the cartesian co-ordinates of its centre of gravity, which may be taken to be ultimately on *OP* at a distance $\frac{2}{3}r$ from O, are $\frac{2}{3}r\cos\theta$ and $\frac{2}{3}r\sin\theta$. Therefore the co-ordinates of the centre of gravity of the sectorial area are

$$\bar{x}=\frac{\int \frac{2}{3}r\cos\theta\,\rho.\frac{1}{2}r^2 d\theta}{\int \rho.\frac{1}{2}r^2 d\theta}=\frac{2}{3}.\frac{\int r^3\cos\theta d\theta}{\int r^2 d\theta},$$

$$\bar{y}=\frac{\int \frac{2}{3}r\sin\theta.\rho.\frac{1}{2}r^2 d\theta}{\int \rho.\frac{1}{2}r^2\,d\theta}=\frac{2}{3}.\frac{\int r^3\sin\theta d\theta}{\int r^2 d\theta},$$

the integrals being taken between proper limits.

When the density ρ is not constant but variable it is very convenient to use double integrals. In the cartesian system, we take as our element the elementary area $dxdy$ at the point (x, y); the mass of the element is $\rho dxdy$ where ρ is the density at the point (x, y) and the co-ordinates of the c. g. of the element are x and y. The general formulæ give the co-ordinates of the c. g. of the area: $\bar{x}=\dfrac{\int\int x.\rho dxdy}{\int\int \rho dxdy}$, $\bar{y}=\dfrac{\int\int y.\rho dxdy}{\int\int \rho dxdy}$, where ρ is to be substituted by its value in terms of x, y.

In the polar system we take as our element the elementary area $rd\theta dr$ at the point (r, θ); the mass of the element is $\rho.rd\theta dr$ where ρ is the density at the point (r, θ) and the cartesian co-ordinates of the c. g. of the element are $r\cos\theta$ and $r\sin\theta$. The general formulæ give the co-ordinates of the centre of gravity of the area:

$$\bar{x}=\frac{\int\int r\cos\theta.\rho rd\theta dr}{\int\int \rho rd\theta dr},\quad \bar{y}=\frac{\int\int r\sin\theta.\rho rd\theta dr}{\int\int \rho rd\theta dr},$$

where ρ is to be substituted by its value in terms of r, θ.

All the above integrals are to be taken over the given area.

5·11. Centre of gravity of any sector of a circle of uniform density.

Let $OACB$ be the given sector, O its centre, a its radius and $\angle AOB=2\alpha$ (fig of Art. 5·8). Let OC bisect the angle AOB and take O as origin and OC as the x-axis. From symmetry the c. g. is on OC. We divide the sector into elementary triangles like OPQ. If $\angle POC=\theta$, area of the element$=\frac{1}{2}a^2d\theta$, and abscissa of its c. g. is $\frac{2}{3}a\cos\theta$.

$$\therefore\quad \bar{x}=\frac{2a}{3}.\frac{\int_{-\alpha}^{\alpha}\cos\theta d\theta}{\int_{-\alpha}^{\alpha}d\theta}=\frac{2a}{3}.\frac{\sin\alpha}{\alpha}.$$

Hence the c. g. of the sector is on the bisector of its angle at a distance from the centre equal to $\frac{2}{3}a\cdot\dfrac{\sin\alpha}{\alpha}$.

When the sector is a quadrant of a circle, we have $2\alpha=\frac{\pi}{2}$ and if $(\bar{x}, \bar{y})$ be the co-ordinates of G, its c. g., referred to OB, OA as axes, we have,

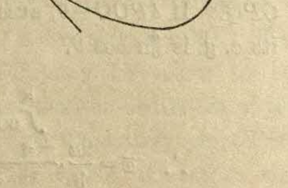
$$\bar{x}=OG.\cos\alpha=\frac{2a}{3}\cdot\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}}\cdot\cos\frac{\pi}{4}=\frac{4a}{3\pi} \text{ and } \bar{y}=\frac{4a}{3\pi}.$$

Hence **the c. g. of a quadrant of a circle is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$,** referred to the bounding radii as axes.

When the sector is a semi-circle $\alpha=\frac{\pi}{2}$ and the c. g. is at a distance $\frac{4a}{3\pi}$ from the centre on the radius perpendicular to its bounding diameter.

5·12. Examples.

Ex. 1. To find the c. g. of a loop of the lemniscate $r^2=a^2\cos 2\theta$.



Here the origin O is the node of the lemniscate and its axis is the axis of x. From symmetry the c. g. is on Ox. We divide the loop into elementary triangles like POP'. If P be (r, θ), the area of the element is $\frac{1}{2}r^2d\theta$ and the x-coordinate of its c. g is $\frac{2}{3}r\cos\theta$.

$$\therefore\quad \bar{x}=\frac{2}{3}\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta}, \text{ as } \theta \text{ varies from } -\frac{\pi}{4} \text{ to } \frac{\pi}{4}$$

$$or,\quad \bar{x}=\frac{2a}{3}\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta d\theta}=\frac{2a}{3}\cdot\frac{\int_{0}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}.\cos\theta d\theta}{\frac{1}{2}},$$

put $\sqrt{2}\sin\theta=\sin t$

$$=\frac{4a}{3}\cdot\frac{1}{\sqrt{2}}\cdot\int_0^{\frac{\pi}{2}}\cos^4 t dt=\frac{4a}{3\sqrt{2}}\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}=\frac{a\pi}{4\sqrt{2}}.$$

Hence the c. g. is on the axis at a distance $\frac{a\pi}{4\sqrt{2}}$ from the node.

Ex. 2. Find the centre of gravity of a semi-circular plate of radius a whose mass per unit area at any point varies as $\sqrt{a^2-r^2}$, where r is the distance of the point from the centre. [C. H. 1963]

Let ACB be the semi-circular plate whose centre is O. Take O as origin and OA as x-axis. Elementary mass at P (r, θ) is $k\sqrt{a^2-r^2}.rd\theta dr$ and co-ordinates of its c. g. are $r\cos\theta$, $r\sin\theta$. From symmetry about y-axis we have the c. g. on the y-axis.

$$\text{And,}\quad \bar{y}=\frac{\int_0^{\pi}\int_0^{a}k\sqrt{a^2-r^2}\,rd\theta dr.r\sin\theta}{\int_0^{\pi}\int_0^{a}k\sqrt{a^2-r^2}.rd\theta dr}$$

$$=\frac{\int_0^{\pi}\sin\theta d\theta\times\int_0^{a}r^2\sqrt{a^2-r^2}dr}{\int_0^{\pi}d\theta\times\int_0^{a}r\sqrt{a^2-r^2}dr}$$

$$= \frac{2.\int_0^{\frac{\pi}{2}} a^4.\sin^2\phi\cos^2\phi.d\phi}{\pi.\left[-\tfrac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^a} \quad \text{putting } r=a\sin\phi\,;$$

or, $\bar{y}=\frac{2}{\pi}\cdot\frac{3a}{8}\cdot\frac{\pi}{2}=\frac{3a}{8}.$

Hence the c. g. is on the radius perpendicular to the bounding diameter at a distance $\frac{3a}{8}$ from the centre.

Ex. 3. Find the centre of gravity of a plate in the form of a quadrant *AOB* of an ellipse, the thickness at any point of the plate varying as the product of the distances of the point from *OA* and *OB*.

Take *OA*, *OB* as axes of x and y and let the thickness at the point (x, y) be $t=kxy$, and let uniform density be ρ. Then mass of element at $(x, y)=\rho.kxydxdy$ and co-ordinates of its c. g. are (x, y) ∴ $(\bar{x}, \bar{y})$ being the c. g of the plate, we have,

$$\bar{x}=\frac{\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \rho kxydxdy.x}{\int\int \rho kxydxdy},$$

[For the quadrant, x varies from 0 to a and when x is x, y varies from 0 to y, where y is given by the equation of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *or,* $y=\frac{b}{a}\sqrt{a^2-x^2}$]

Cancelling ρk,

$$\text{Numerator}=\int_0^a x^2.\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx=\frac{1}{2}\cdot\frac{b^2}{a^2}\int_0^a x^2(a^2-x^2)dx$$

$$=\frac{b^2}{2a^2}\cdot\tfrac{2}{15}a^5=\tfrac{1}{15}.b^2a^3.$$

$$\text{Denominator}=\frac{1}{2}\cdot\frac{b^2}{a^2}\int_0^a x(a^2-x^2)dx=\frac{b^2}{2a^2}\cdot\frac{a^4}{4}=\tfrac{1}{8}b^2a^2.$$

∴ $\bar{x}=\tfrac{8}{15}a.$ Similarly $\bar{y}=\tfrac{8}{15}b.$

Hence the c. g. is $(\tfrac{8}{15}a, \tfrac{8}{15}b)$.

Let the thickness or density at the point (x, y) be $kx^m y^n$.

$$\text{Then}\quad \bar{x}=\frac{\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kx^m y^n.dxdy.x}{\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kx^m y^n.dxdy}$$

$$=\frac{\int_0^a x^{m+1}(a^2-x^2)^{\frac{n+1}{2}}dx}{\int_0^a x^m(a^2-x^2)^{\frac{n+1}{2}}dx}.$$

To integrate put $x=a\sin\theta$

$$\therefore\quad \bar{x}=\frac{a\int_0^{\frac{\pi}{2}}(\sin\theta)^{m+1}.(\cos\theta)^{n+2}d\theta}{\int_0^{\frac{\pi}{2}}(\sin\theta)^m.(\cos\theta)^{n+2}d\theta}$$

$$=a\cdot\frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{m+n+5}{2}\right)}\times\frac{\Gamma\left(\frac{m+n+4}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)},\ \text{using Gamma function.}$$

Similarly $\bar{y}$ can be found. $\bar{y}$ can be found from $\bar{x}$ by writing b for a and interchanging m, n.

Ex. 4. The density at any point of a circular lamina varies as the nth power of the distance from a point O on the circumference ; show that the centre of gravity of the lamina divides the diameter through O in the ratio $(n+2) : 2$.

Take O as origin and the diameter OA as the initial line and let $P\ (r, \theta)$ be any point of the lamina. The elementary area at P is $rd\theta dr$ and $\rho=kr^n$ (fig. Art 5'18).

$\therefore$ Elementary mass $dm=rd\theta dr.kr^n$ and the co-ordinates of the c. g. of the elementary mass are $r\cos\theta$, $r\sin\theta$. From symmetry the c. g. is on OA.

$$\therefore\ \bar{x}=\frac{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\limits_{0}^{2a\cos\theta} kr^{n+1}d\theta dr.r\cos\theta}{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\limits_{0}^{2a\cos\theta} kr^{n+1}d\theta dr},\ \text{where } a=\text{radius of the circle.}$$

$$=2a\cdot\frac{n+2}{n+3}\cdot\frac{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\,\theta)^{n+4}d\theta}{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\,\theta)^{n+2}d\theta}$$

$$=2a\cdot\frac{n+2}{n+3}\cdot\frac{\Gamma\left(\frac{n+5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\times\frac{\Gamma\left(\frac{n+4}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$=2a\frac{n+2}{n+3}\cdot\frac{n+3}{2}\cdot\frac{1}{\frac{n+4}{2}}=2a\cdot\frac{n+2}{n+4}.$$

Hence the c. g. divides OA in the ratio $(n+2):2$.

EXAMPLES

1. Find the centres of gravity of the following areas :

(a) a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(b) a quadrant of the four-cusped hypocycloid $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

(c) the area included by a parabola and two tangents. (The eqn of the parabola referred to the tangents as axes is $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$).

(d) a semi-ellipse cut off by any diameter.

(e) any segment of a parabola cut off by a straight line.

(f) the area between the curve $y^2(2a-x)=x^3$ and its asymptote.

(g) the area between the catenary $y=c\cosh\frac{x}{c}$, the co-ordinate axes and the ordinate $x=a$.

2. Find the centres of gravity of :

(*a*) the area of a semi-cardioid.

(*b*) the area of a semi-cycloid.

(*c*) the area of the half of a loop of the lemniscate.

(*d*) the area of a loop of $r=a \cos 2\theta$.

(*e*) a segment of a circle.

3. Find the centroid of the area enclosed between the circles $x^2+y^2-2ax=0$ and $x^2+y^2-2bx=0$ and lying above the axis of x.

4. Find the centroid of the area included between the arc of a semi-cycloid, the generating circle and the base of the cycloid.

5. Show that the co-ordinates of the centroid of the portion of an ellipse cut off by a line joining the extremities of the two axes are $\bar{x}=\frac{2}{3}\cdot\frac{a}{\pi-2}$, $\bar{y}=\frac{2}{3}\cdot\frac{b}{\pi-2}$.

6. Find the centroid of a quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$, the density at the point (x, y) being (1) kxy, (2) ky, (3) kx^2y^2.

7. Show that the centroid of a circular sector in which the density varies as the nth power of the distance from the centre is on the axis of the sector at a distance $\frac{n+2}{n+3}.a.\frac{\sin\alpha}{\alpha}$, where 2α is the angle of the sector.

8. A circular disc of radius r, whose density varies as the distance from the centre, has a hole cut in it bounded by a circle of diameter a which passes through the centre. Show that the distance from the centre of the disc of the centre of gravity of the remaining portion is $\frac{6a^4}{15\pi r^3-10a^3}$.

5·13. Centre of gravity of a surface of revolution.

The given matter is distributed in the form of a portion of a surface of revolution generated by revolving a plane finite curve BC about a line in its plane which we take as the x-axis Ox. (fig. of Art. 5·10). Due to the revolution, every point of the curve describes a circle whose centre is on Ox. Let P be any point on the arc whose co-ordinates are x, y and let s be the arcual distance of P measured from some fixed point on the curve. The elementary arc PP' of length ds, by revolving about Ox, generates an elementary portion of the surface in the form of a narrow circular band whose area is $2\pi y ds$. If ρ be the density at P, the mass of the elementary circular band is $dm=\rho.2\pi y ds$ and its centre of gravity is on Ox at a distance x from O. Therefore the c. g. of the surface is on the axis of revolution Ox and at a distance from O equal to $\bar{x}=\frac{\int x\,\rho 2\pi y ds}{\int \rho.2\pi y ds}=\frac{\int \rho x y ds}{\int \rho y ds}$, the integrals being taken between proper limits.

If we use polar co-ordinates, we should write $x=r\cos\theta$, $y=r\sin\theta$ in the above integrals and substitute ds by its value $\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta$ and integrate between proper limits of θ. If ρ be a constant, it may be omitted from the formulæ.

5·14. Centre of gravity of a zone of the surface of a sphere.

Let AB be an arc of a circle whose centre is O. Let OC be the diameter such that by revolving the arc AB about OC the given zonal surface is generated. We take O as origin and OC as x-axis. Let a be the radius of the circle and let $\angle COA=\alpha$ and $\angle COB=\beta$. We have $ds=a\,d\theta$, $x=a\cos\theta$, $y=a\sin\theta$. Using the results obtained in the previous article we have

$$\bar{x}=\frac{\int_{\alpha}^{\beta}a^2\sin\theta\cos\theta\,a\,d\theta}{\int_{\alpha}^{\beta}a\sin\theta.a\,d\theta}=a.\tfrac{1}{2}\frac{(\cos^2\alpha-\cos^2\beta)}{(\cos\alpha-\cos\beta)}$$

$$=\tfrac{1}{2}.(a\cos\alpha+a\cos\beta)=\tfrac{1}{2}(OL+OM),$$

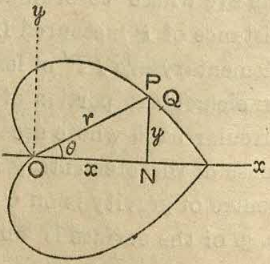
where AL, BM are perpendiculars from A and B on OC.

= middle point of LM.

Hence the c. g. of a zone of a spherical surface is on the axis of the zone and *mid way between the bounding planes.*

In particular, **the c. g. of a hemispherical shell is the middle point of the radius perpendicular to its plane face.**

5·15. Ex. 1. To find the c. g. of the surface formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis.



Here the cusp is the origin and the axis of the cardioid is the axis of x. Differentiating the equation $r=a(1+\cos\theta)$ with respect to θ, we have $\frac{dr}{d\theta}=-a\sin\theta$.

$$\therefore\quad ds=\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}d\theta$$

$$=[a(1+\cos\theta)^2+a^2\sin^2\theta]^{\frac{1}{2}}d\theta=2a.\cos\frac{\theta}{2}d\theta.$$

From symmetry, the c.g of the surface is on Ox and its distance from O is

$$\bar{x}=\frac{\int xyds}{\int yds}, \text{ using result obtained in Art. 5·13.}$$

$$=\frac{\int_0^\pi r^2 \sin\theta\cos\theta . 2a\cos\frac{\theta}{2}d\theta}{\int_0^\pi r\sin\theta . 2a\cos\frac{\theta}{2}d\theta}=\frac{2a\int_0^\pi \cos^5\frac{\theta}{2}\sin\theta\cos\theta d\theta}{\int_0^\pi \cos^3\frac{\theta}{2}\sin\theta d\theta}$$

$$=\frac{2a\int_0^\pi \cos^6\frac{\theta}{2}\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta}{\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta}$$

$$=\frac{2a\cdot\left[-\frac{4}{9}\cos^9\frac{\theta}{2}+\frac{2}{7}\cos^7\frac{\theta}{2}\right]_0^\pi}{\left[-\frac{2}{5}\cos^5\frac{\theta}{2}\right]_0^\pi}=2a\cdot\frac{\frac{10}{63}}{\frac{2}{5}}=\frac{50}{63}a.$$

Hence the c. g. is on the axis at a distance $\frac{50a}{63}$ from the cusp.

Ex. 2. Show that the centre of gravity of a lune of a sphere, of angle 2α, is at a distance $\frac{\pi}{4}\cdot\frac{a\sin\alpha}{\alpha}$ from the axis.

The lune is the surface generated by the arc of a semi-circle of radius a when it revolves about the bounding diameter through the angle 2α. Let AOB be the bounding diameter and O the centre. Take O as origin and OA as x-axis. The length of the element of arc at P where $\angle AOP=\theta$ is $ad\theta$ and by revolving about AOB through an angle 2α it generates a circular band of breadth $ad\theta$ and length $2\alpha.a\sin\theta$.

$\therefore$ Elementary mass $=\rho.ad\theta.2\alpha.a\sin\theta$ and its c. g. is at a distance $a\sin\theta\cdot\frac{\sin\alpha}{\alpha}$ from AOB (Art. 5·8).

$\therefore$ Distance of the c. g. of the lune from AOB

$$=\frac{\int_0^{\pi} \rho.ad\theta.2\alpha.a\sin\theta.a\sin\theta.\frac{\sin\alpha}{\alpha}}{\int_0^{\pi} \rho\, ad\theta.2\alpha.a\sin\theta}$$

$$=a.\frac{\sin\alpha}{\alpha}.\frac{\frac{1}{2}\pi}{2}=\frac{\pi}{4}.a.\frac{\sin\alpha}{\alpha}.$$

5·16. Centre of gravity of a solid of revolution.

The given matter is distributed in the form of a portion of a solid of revolution generated by revolving a plane area $ABCD$ about a line in its plane which we take as Ox, the axis of x (fig. Art. 5·10). The area $ABCD$ is bounded by the ordinates at A and D, the x-axis OA, and the curve BC whose equation is given. When the density ρ is constant, we divide the area into thin strips like $PP'Q'Q$, whose sides are parallel to the y-axis. The elementary area $PP'Q'Q$ by revolving about Ox generates an elementary portion of the solid in the form of a thin circular disc of area πy^2 and thickness dx, whose mass dm is therefore $\rho\pi y^2dx$, where P is (x, y). The centre of gravity of this elementary circular disc is ultimately at Q, whose co-ordinates are $(x, 0)$. Therefore the c. g. of the solid is given by

$$\bar{x}=\frac{\int x\,\rho\pi y^2dx}{\int \rho\pi y^2dx}=\frac{\int xy^2dx}{\int y^2dx},\quad \bar{y}=0.$$

When the density ρ is not constant but variable we may conveniently use double integrals. If we use the cartesian system, we take a small elementary area $dxdy$ at a point $P\,(x, y)$ of the area. By revolving about Ox, it generates an elementary uniform circular ring of cross-section $dxdy$ whose centre is on Ox and whose radius is y. The mass of this element of volume is $dm=\rho.2\pi y.dxdy$, where ρ is the density at P, and its centre of gravity is on Ox at a distance x from O. Therefore the c. g. of the solid is on Ox, the axis of revolution and its distance from O is $\bar{x}=\dfrac{\iint x.\rho 2\pi ydxdy}{\iint \rho.2\pi ydxdy}=\dfrac{\iint \rho xydxdy}{\iint \rho ydxdy}$. If we use the polar system, then the elementary area at $P\,(r, \theta)$ is $rd\theta dr$ and so, in this case, $dm=\rho.2\pi yrd\theta dr=\rho.2\pi r\sin\theta.rd\theta dr$ and the c. g. of the element is on Ox at a distance $x=r\cos\theta$ from O.

$\therefore$ The c. g. of the solid is on Ox and

$$\bar{x}=\frac{\int\int \rho.2\pi r^2 \sin\theta d\theta dr\, r\cos\theta}{\int\int \rho 2\pi r^2 \sin\theta d\theta dr}=\frac{\int\int \rho r^3 \sin 6 \cos\theta d\theta dr}{\int\int \rho . r^2 \sin\theta d\theta dr}.$$

5·17. Centre of gravity of a solid sector of a sphere.

Suppose the solid sector is generated by revolving the sectorial area OAC (fig. Art. 5·8) about OC. We take O as origin and Ox as x-axis. Let a be the radius of the circular sector and let $\angle COA=\alpha$. Using polar co-ordinates and the result obtained in previous article, we find that the c. g. of the solid sector is on OC at a distance from O equal to

$$\bar{x}=\frac{\int_0^a\int_0^\alpha r^3 \sin\theta\cos\theta d\theta dr}{\int_0^a\int_0^\alpha r^2\sin\theta d\theta dr}=\frac{3}{4}a\cdot\frac{1}{2}(1+\cos\alpha).$$

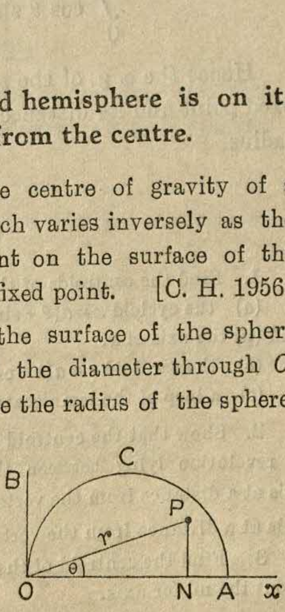
In particular, if the sector be a solid hemisphere, we have

$$\alpha=\frac{\pi}{2}, \text{ and } \therefore\ \bar{x}=\frac{3}{8}a.$$

Hence **the c. g. of a uniform solid hemisphere is on its axis of symmetry at a distance $\frac{3}{8}$a from the centre.**

5·18. Example. Show that the centre of gravity of a sphere, the density at any point of which varies inversely as the square of the distance from a fixed point on the surface of the sphere, bisects the radius through the fixed point. [C. H. 1956]

Let O be the the fixed point on the surface of the sphere which is taken as the origin, and OA the diameter through O, which is taken as the x-axis. Let a be the radius of the sphere. The sphere is a solid of revolution generated by revolving about OA the semi-circle OCA, drawn on OA as diameter. Let P (r, θ) be any point of the semi-circle, the density at P is $\frac{k}{r^2}$. The element of area



$rd\theta dr$ at P, by revolution about OA, generates a ring of radius $PN=r\sin\theta$ and cross-section $rd\theta dr$ and whose centre is N.

∴ The element of mass thus generated is

$rd\theta dr.2\pi r\sin\theta\cdot\frac{k}{r^2}=2\pi k\sin\theta drd\theta$ and its c. g. is N whose co-ordinates are $(r\cos\theta, 0)$.

∴ The c. g. of the solid is on OA and its distance from O is

$$\bar{x}=\frac{\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\sin\theta drd\theta.r\cos\theta}{\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\sin\theta drd\theta}$$

$$=a\frac{\int_0^{\frac{\pi}{2}}\cos^3\theta\sin\theta d\theta}{\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta}=a\cdot\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}a.$$

Hence the c. g. of the solid is on the diameter through the fixed point and its distance from the latter is one-half of the radius.

EXAMPLES

1. Find the centroids of the surfaces formed by the revolution of :

(*a*) the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about (*i*) the axis of y, and (*ii*) the axis of x.

(*b*) the cardioid $r=a(1+\cos\theta)$ about its axis.

(*c*) a loop of the lemniscate $r^2=a^2\cos 2\theta$ about the initial line.

2. Show that the centroid of the portion of the surface of a paraboloid of revolution lying between the vertex and a plane perpendicular to the axis at a distance from the vertex equal to $\frac{3}{4}$ of the latus rectum is on the axis at a distance from the vertex equal to $\frac{29}{70}$ (latus rectum).

3. Find the centroid of the surface of a semi-ellipsoid of revolution about the minor-axis.

4. A uniform bowl has for its inner and outer surfaces concentric hemispheres of radii a and b respectively ; prove that the distance of its c. g. from the centre is $\frac{3(a+b)(a^2+b^2)}{8(a^2+ab+b^2)}$.

5. Find the centroids of the volumes formed by the revolution of :

(*a*) the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about (*i*) the axis of y and (*ii*) the axis of x.

(*b*) the cardioid $r=a(1+\cos\theta)$ about its axis.

(*c*) a semi-ellipsoid of revolution about minor axis cut off by a plane through the axis when (*i*) the density is uniform (*ii*) the density at a point varies as its distance from the plane $x=0$.

6. Find the centroid of the solid segment of a sphere bounded by a plane distant c from the centre.

7. Find the centroid of a sphere whose density varies as the distance from a tangent plane.

8. Prove that in a sphere, whose density varies inversely as the distance from a point in the surface, the distance of the c. g. from that point bears to the diameter the ratio 2 : 5. [M. T.]

9. A uniform solid formed by revolution of a cycloid about its base is cut in halves by a plane through the axis. Show that the centroid of a half is at a distance $\frac{7a}{3\pi}$ from the plane face.

5·19. Centre of gravity of any solid.

The fundamental formulae to be used are those given in Art. 5·6. For cartesian co-ordinates we take $dm=\rho dxdydz$ and the integration is a triple integration.

Thus we have $\bar{x}=\frac{\int\int\int x.\rho dxdydz}{\int\int\int \rho dxdydz}$, with similar expressions for $\bar{y}$ and $\bar{z}$.

For polar co-ordinates, we take $dm=\rho.r^2\sin\theta drd\theta d\phi$,

$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$, and the integration is a triple integration.

Thus we have $\bar{x}=\frac{\int\int\int \rho.r^3\sin^2\theta\cos\phi\, drd\theta d\phi}{\int\int\int \rho r^2\sin\theta\, drd\theta d\phi}$, etc.

5·20. Example :

Find the c. g. of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ bounded by the principal planes when the density at a point (x, y, z) is $kxyz$.

Consider the octant in which $x, y, z>0$.

We have, $dm=kxyz.dxdydz$.

$$\therefore \quad \bar{x}=\frac{\iiint_V xkxyz\,dx\,dy\,dz}{\iiint_V kxyz\,dx\,dy\,dz},$$ where V denotes the given octant of the ellipsoid.

To integrate put $x=aX$, $y=bY$, $z=cZ$.

$$\therefore \quad \bar{x}=\frac{a\iiint_{V'} X^2YZ\,dX\,dY\,dZ}{\iiint_{V'} XYZ\,dX\,dY\,dZ},$$ where V' denotes the corresponding octant of the sphere $X^2+Y^2+Z^2=1$.

Now we make the polar transformation,

$$X=r\sin\theta\cos\phi,\ Y=r\sin\theta\sin\phi,\ Z=r\cos\theta$$

$$\therefore \quad J\left(\frac{X,\,Y,\,Z}{r,\,\theta,\,\phi}\right)=r^2\sin\theta.$$

$$\therefore \quad \bar{x}=\frac{a.\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} d\phi.r^6.\sin^4\theta.\cos\theta.\cos^2\phi\sin\phi}{\int_0^1 dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} d\phi\, r^5.\sin^3\theta.\cos\theta\cos\phi\sin\phi}$$

$$=\frac{a.\frac{1}{7}.\frac{1}{5}.\frac{1}{3}}{\frac{1}{6}.\frac{1}{4}.\frac{1}{2}}=\frac{16a}{35}.$$

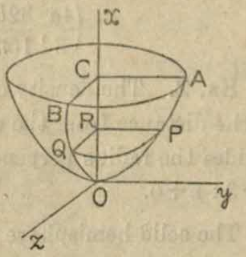
Similarly $\bar{y}=\dfrac{16b}{35}$ and $\bar{z}=\dfrac{16c}{35}$.

Hence the c. g. of the octant is $\left(\dfrac{16a}{35},\dfrac{16b}{35},\dfrac{16c}{35}\right)$.

5·21. Instead of using the general formulae with triple integrations it is sometimes more advantageous to divide the solid, if possible, into larger elements like thin slices or shells or if the solid be heterogeneous, thin strata of homogenous matter, whose masses and centres of gravity are known. The c. g. may be found by a single integration. We work out two examples to illustrate the method.

Ex. 1. Find the centroid of the volume included between the surfaces $\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-\dfrac{2x}{a}=0$, $x=2a$, $y=0$ and $z=0$. [C. H. 1951]

The first surface is an elliptic paraboloid whose axis is Ox. The others are planes. The given volume is $ACBO$ where $OC=2a$ and AOC and BOC are sections by planes $z=0$ and $y=0$ respectively. We divide the volume into thin discs which are quadrants of ellipses by drawing planes parallel to the yOz plane. Consider one such disc PRQ where $OR=x$.



The equation of the section by the plane $x=x$ is $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{2x}{a}$

$$or,\quad \frac{y^2}{\frac{2b^2x}{a}}+\frac{z^2}{\frac{2c^2x}{a}}=1. \quad \therefore \quad RP=\sqrt{\frac{2b^2x}{a}},\ RQ=\sqrt{\frac{2c^2x}{a}}.$$

$$\therefore \quad \text{Vol. of the disc}=\tfrac{1}{4}\pi.\sqrt{\frac{2b^2x}{a}\cdot\frac{2c^2x}{a}}=\frac{\pi}{4}\cdot\frac{2}{a}bcx=\frac{\pi bc}{2a}\cdot x$$

Co-ordinates of its c. g. are $x=x$, $y=\frac{4}{3\pi}\sqrt{\frac{2b^2x}{a}}$, $z=\frac{4}{3\pi}\sqrt{\frac{2c^2x}{a}}$

$$\therefore \quad \bar{x}=\frac{\int_0^{2a}\frac{\pi bc}{2a}x.xdx}{\int_0^{2a}\frac{\pi bc}{2a}\cdot xdx}=\frac{2}{3}\cdot 2a=\frac{4a}{3}.$$

$$\bar{y}=\frac{\int_0^{2a}x\cdot\frac{4}{3\pi}\sqrt{\frac{2b^2x}{a}}dx}{\int_0^{2a}xdx}=\frac{\frac{4b}{3\pi}\cdot\sqrt{\frac{2}{a}}\cdot\frac{2}{5}(2a)^{\frac{5}{2}}}{\frac{(2a)^2}{2}}=\frac{32b}{15\pi}.$$

$$\text{Similarly } \bar{z}=\frac{\int_0^{2a}x\cdot\frac{4}{3\pi}\cdot\sqrt{\frac{2c^2x}{a}}dx}{\int_0^{2a}xdx}=\frac{32c}{15\pi}.$$

Hence the centre of gravity of the given volume is

$$\left(\frac{4a}{3}, \frac{32b}{15\pi}, \frac{32c}{15\pi}\right).$$

Ex. 2. The density of a hemisphere varies as the nth power of the distance from the centre ; show that the centre of gravity divides the radius perpendicular to its plane surface in the ratio $n+3 : n+5$. [C. H. 1960, 1958]

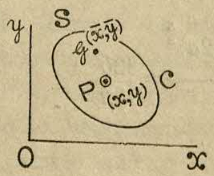
The solid hemisphere can be broken up into thin homogeneous hemispherical shells of radii varying from 0 to a, the radius of the given solid hemisphere, all having the same centre O as that of the solid, by drawing concentric spheres of centre O. The mass of the shell lying between spheres of radii r and $r+dr$ is $2\pi r^2.dr.kr^n$ and its c. g. is on the radius of the hemisphere perpendicular to the plane face at a distance $\frac{r}{2}$ from O. Therefore, the c. g. G of the solid is on this radius and OG

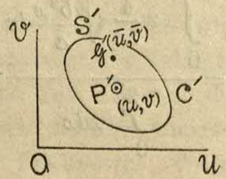
$$=\frac{\int_0^a 2\pi kr^{n+2}.\frac{r}{2}dr}{\int_0^a 2\pi kr^{n+2}\,dr}=\frac{1}{2}\cdot\frac{a}{n+4}\cdot\frac{n+3}{1}=\frac{n+3}{2n+8}\cdot a.$$

Hence the c. g. of the solid is on the radius perpendicular to its plane face and divides the radius in the ratio $(n+3):(n+5)$.

5·22. Transformation or change of co-ordinates.

Sometimes transformation may be used advantageously to find the c. g. of a plane area or of a volume.





The simple transformation $x=au$, $y=bv$, establishes a one-one correspondence between the points of the xy and the uv-planes. A point P or a curve C or a region S in the xy-plane is transformed into a point P' or a curve C' or a region S' in the uv-plane. These are called corresponding elements. Suppose in the xy-plane we have a region S bounded by a coutour C and suppose A is the area and $(\bar{x}, \bar{y})$ is the c. g. of S. Let S' be the corresponding region in the uv-plane bounded by the corresponding coutour C' and let A' be the area and $(\bar{u}, \bar{v})$ the c. g. of S'.

We have $A=\iint_S dxdy=\iint_{S'} abdudv=ab.A'$. Thus the ratio of the areas of corresponding regions is constant.

Again, we have, $\bar{x}=\dfrac{\iint_S xdxdy}{A}=\dfrac{\iint_{S'} a^2b.ududv}{abA'}=a\bar{u}.$

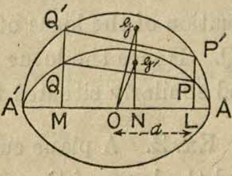
Similarly $\bar{y}=b\bar{v}$.

Thus the c. g's of corresponding regions are corresponding points. The orthogonal projection of a plane area on another plane is a particular case of this transformation, where $x=u \sec \alpha$, $y=v$, α being the angle between the planes.

We have exactly identical results in three dimensions for the transformation $x=au$, $y=bv$, $z=cw$.

These transformations are useful in finding the c. g's of elliptic areas or ellipsoidal volumes from those of circles and spheres. We shall use it to solve the following problems.

Ex. 1. A right line cuts off a constant area from an ellipse; find the locus of the centroid of the portion cut off.



Let the ellipse be $APQA'$ whose centre is O and major axis $A'OA$. Let chord PQ cut off a constant area from the ellipse.

Equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \cdots\cdots(1)$$

We make the transformation $x=X$, $y=\frac{b}{a}Y$. $\cdots(2)$

Then the ellipse transforms into the auxiliary circle

$$X^2+Y^2=a^2 \ldots\ldots(3)$$

The transformation (2) transforms the chord PQ into the corresponding chord $P'Q'$ of the circle (3) and the segment cut off by PQ into the corresponding segment cut off by $P'Q'$.

Area of segment cut off by chord PQ from the ellipse

$$=\int\int dxdy=\int\int dX\cdot\frac{b}{a}dY=\frac{b}{a}\int\int dXdY$$

$$=\frac{b}{a}\times \text{area cut off by chord } P'Q' \text{ from the circle.}$$

$\therefore$ When PQ cuts off a constant area from the ellipse (1) $P'Q'$ also cuts off a constant area from the circle (3).

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the c. g of the segment cut off by PQ and let $(\bar{X}, \bar{Y})$ be the c. g. of the corresponding segment cut off by $P'Q'$.

$$\text{Then } \bar{x}=\frac{\int\int xdxdy}{\int\int dxdy}=\frac{\int\int X.dXdY}{\int\int dXdY}=\bar{X}$$

$$\text{and } \bar{y}=\frac{\int\int ydxdy}{\int\int dxdy}=\frac{b}{a}\frac{\int\int YdXdY}{\int\int dXdY}=\frac{b}{a}\bar{Y}.$$

Thus the c. g's are also correspoding points with respect to the transformation (2).

Now, in the case of the circle, as the area of the segment cut off by $P'Q'$ is constant, $(\bar{X}, \bar{Y})$, the c. g. of the segment, which lies on the line through O perpendicular to $P'Q'$, is at a constant distance from O. Hence the locus of $(\bar{X}, \bar{Y})$ is a circle whose centre is O. Let the equation of this circle be $\bar{X}^2+\bar{Y}^2=c^2$. Since $\bar{X}=\bar{x}$ and $\bar{Y}=\frac{a}{b}\bar{y}$ we have, $\bar{x}^2+\frac{a^2}{b^2}\bar{y}^2=c^2$; this is the equation of the locus of $(\bar{x}, \bar{y})$, the c. g. of the segment cut off by PQ. Hence the locus required is a concentric ellipse, similar and similarly situated to the given ellipse.

Ex. 2. A plane cuts off a constant volume from an ellipsoid ; find the locus of the centroid of the portion cut off.

Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \cdots \quad (1)$

We make the transformation $x=aX,\ y=bY,\ z=cZ. \quad \cdots \quad (2)$

The ellipsoid (1) is transformed into the sphere

$$X^2+Y^2+Z^2=1 \quad \cdots \quad (3)$$

It can be shown similarly as in previous example, that the transformation (2) transforms equal volumes into equal volumes and that the c. g.'s of corresponding volumes are corresponding points. The result easily follows as in the above example. The required locus is an ellipsoid concentric and similar and similarly situated to the given ellipsoid.

5·23. The more general transformation $x=\phi(u, v)$, $y=\psi(u, v)$ is sometimes used to simplify double integrations. The transformed integral is given by the formulæ,

$\int\int_R f(x, y)dxdy=\int\int_R f(\phi, \psi) \mid J \mid dudv$. In three dimensions we have $x=\phi(u, v, w)$, $y=\psi(u, v, w)$, $z=\chi(u, v, w)$ and

$$\int\int\int_R f(x, y, z)dxdydz=\int\int\int_R f(\phi, \psi, \chi) \mid J \mid dudvdw,$$

where J is the Jacobian. These results are to be found in treatises on the Integral Calculus. This method has been used in solving the example in Art. 5·20. We now work out an example of a different type.

Example

Find the c. g. of the area enclosed by the four parabolas $y^2=4ax$, $y^2=4bx$, $x^2=4cy$, $x^2=4dy$.

Here the figure formed is a curvilinear quadrilateral whose sides are arcs of the given parabolas. If $(\bar{x}, \bar{y})$ be the c. g. of the area enclosed, we have,

$\bar{x}=\dfrac{\int\int xdxdy}{\int\int dxdy}$. $\bar{y}=\dfrac{\int\int ydxdy}{\int\int dxdy}$, where each of the integrations is to be taken over the area enclosed. To simplify the calculation, we make the transformation given by $y^2=4ux$ and $x^2=4vy$. Solving $x=4u^{\frac{1}{3}}v^{\frac{2}{3}}$ and $y=4u^{\frac{2}{3}}v^{\frac{1}{3}}$, which gives x, y in terms of u, v.

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)}=\begin{vmatrix} \dfrac{\partial x}{\partial u}. & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u}. & \dfrac{\partial y}{\partial v} \end{vmatrix}=-\frac{48}{9} \quad \therefore \quad \mid J \mid=\frac{48}{9}.$$

and the enclosed area has been transformed into a rectangle bounded by the lines $u=a$, $u=b$; $v=c$, $v=d$.

$$\therefore \quad \bar{x}=\frac{\dfrac{48}{9}\int_a^b du\int_c^d dv.4u^{\frac{1}{3}}v^{\frac{2}{3}}}{\dfrac{48}{9}\int_a^b du\int_c^d dv}=\frac{9}{5}\cdot\frac{(b^{\frac{4}{3}}-a^{\frac{4}{3}})(d^{\frac{5}{3}}-c^{\frac{5}{3}})}{(b-a)(d-c)}$$

similarly, $\bar{y}=\frac{9}{5}\cdot\frac{(b^{\frac{5}{3}}-a^{\frac{5}{3}})(d^{\frac{4}{3}}-c^{\frac{4}{3}})}{(b-a))d-c)}$

5·24. Theorems of Pappus.

The two following theorems were first given by Pappus. With their help we can find the surface area and the volume of a solid of revolution when we know the centres of gravity of the generating curve and of the generating area.

Theorem I.

If an arc of a plane curve rotates through any angle about an axis in its plane which does not cross it, then the area of the surface generated by the revolving arc is equal to the length of the arc multiplied by the length of the path described by its centre of gravity.

Let BC be an arc of a plane curve which lies in the plane xOy and let Ox be the axis of revolution which does not cut the curve (fig. Art. 5·10). Any elementary arc $PP'=ds$, where P is (x, y), rotating round Ox through an angle θ traces out an elementary area in the form of a narrow band of length $y\theta$ and breadth ds, whose area is therefore $y\theta ds$. $\therefore$ area of the whole surface generated by the arc $BC=\int y\theta ds=\theta\int yds$. But if $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of the arc BC, we have $\bar{y}=\frac{\int yds}{\int ds}$ $\therefore$ $\int yds=\bar{y}.s$, where s is the length of the arc BC. Hence the area of the surface generated $=\theta.\bar{y}s.=s.\bar{y}\theta=$ length of the arc $BC\times$length of the path described by the centre of gravity of the arc BC.

Theorem II.

If a plane area rotates through any angle about an axis in its plane which does not cross it, then the volume generated by the revolving area is equal to the area multiplied by the length of the path described by its centre of gravity.

Let any given plane area lie in the plane xOy and let Ox be the axis of revolution which does not cut the area (fig. Art. 5·10). Any elementary area dA at a distance y from Ox, rotating round Ox through an angle θ, traces out an elementary volume in the form of a narrow tube of length $y\theta$ and cross-section dA, whose volume is therefore $y\theta dA$. $\therefore$ Volume generated by the given area $=\int y\theta dA=\theta\int ydA$. But if $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of the given area, we have, $\bar{y}=\dfrac{\int ydA}{\int dA}$.

$\therefore$ $\int ydA=\bar{y}.A$, where A is the given area. $\therefore$ Volume generated by the given area $=\theta.\bar{y}A=A.\bar{y}\theta=$ given area $\times$ length of the path described by the centre of gravity of the given area.

For example, the surface of a sphere is generated by revolving a semi-circular arc through an angle 2π, round its bounding diameter. Now length of semi-circular arc $=\pi r$, r being the radius. Distance of the centre of gravity of the semi-circular arc from the diameter $=\dfrac{2r}{\pi}$. $\therefore$ Length of the path described by the centre of gravity $=\dfrac{2r}{\pi}\times 2\pi=4r$.

$\therefore$ Area of the surface of the sphere $=\pi r.4r=4\pi r^2$.

Again, a solid sphere is generated by revolving a semi-circular area through an angle 2π round the bounding diameter. Area of the semi-circle is $\frac{1}{2}\pi r^2$. Distance of its centre of gravity from the diameter is $\dfrac{4r}{3\pi}$.

$\therefore$ Volume of the sphere $=\frac{1}{2}\pi r^2\times\left(\dfrac{4r}{3\pi}\times 2\pi\right)=\dfrac{4}{3}\pi r^3$.

5·25. Lagrange's Theorem.

If particles of masses $m_1, m_2, \ldots\ldots$ be situated at the points $A_1, A_2, \ldots\ldots$ and if G be their centre of gravity and O be any other point in space, then

$$\Sigma m.OA^2=\Sigma m.GA^2+(\Sigma m.)OG^2.$$

We take G as origin and any rectangular system of axes. Let $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots\ldots$ be the co-ordinates of $A_1, A_2 \ldots\ldots$ and let O be the point (ξ, η, ζ). Then since the origin is the c. g. of the masses $m_1, m_2, \ldots\ldots$ placed at $A_1, A_2, \ldots\ldots$ we have,

$$\frac{\Sigma mx}{\Sigma m}=0 \quad i.e. \quad \Sigma mx=0. \quad \text{Similarly } \Sigma my=0 \text{ and } \Sigma mz=0 \ldots(1)$$

$$\begin{aligned}\text{Now } \Sigma m.OA^2&=\Sigma m.[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]\\&=\Sigma m(x^2+y^2+z^2)-2\xi\Sigma mx-2\eta\Sigma my-2\zeta\Sigma mz\\&\qquad+\Sigma m(\xi^2+\eta^2+\zeta^2)\\&=\Sigma m.GA^2+(\Sigma m).OG^2, \quad \text{[using relations (1)].}\\&=\Sigma m.GA^2+M.OG^2.\end{aligned}$$

where $M=\Sigma m=$ Total mass.

EXAMPLES

1. Find the centroid of the volumes included by the surfaces :

(*a*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0,\ lx+my+nz=1.$

(*b*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ x=0,\ z=\pm c.$

(c) $2x^2+y^2=2ax,\ z=mx,\ z=nx.$

2. Find the centre of gravity of an octant of the ellipsoid when the density (1) ρ is constant (2) $\rho=kx^l y^m z^n$ and (3) $\rho\propto$ the square of the distance from the centre.

3. Find the centroid of a hemisphere when the density varies as the distance from a point on its plane edge.

4. Find the centroid of an octant of the solid $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+\left(\frac{z}{c}\right)^n=1.$

5. Find the volume and surface of an anchor-ring.

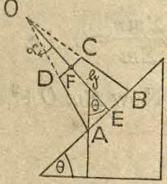
6. A triangle of area Δ revolves about a straight line in its own plane the perpendiculars on which from its angular points are p_1, p_2, p_3 ; show that the volume generated is $\frac{2\pi}{3}\Delta(p_1+p_2+p_3)$.

7. The lemniscate $r^2=a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$.

5·26. Miscellaneous Problems worked.

Ex. 1. A frustum of a solid right cone is placed with its base on a rough inclined plane whose inclination is gradually increased ; if R and r be the radii of the larger and smaller sections, and h be the height of the frustum, show that the frustum will ultimately either tumble or slide, according as the coefficient of friction $\gtrless\frac{4R}{h}\cdot\frac{R^2+Rr+r^2}{R^2+2Rr+3r^2}$. [C. H. 1960]

We know the frustum will slide only when the inclination of the plane has increased to λ, the angle of friction. Let θ be the inclination when the frustum tumbles. Then the frustum will tumble or slide according as $\theta\lesseqgtr\lambda$. Let $ABCD$ be the frustum, G its c. g. which is on the axis OFE of the cone OAB. Since the frustum is about to tumble the vertical through G passes through A, the lowest point of its base AB in contact with the plane.



$$\therefore\ \angle AGE=\theta \text{ and } \tan\theta=\frac{AE}{EG}=\frac{R}{EG}\quad\cdots\quad\cdots\quad\cdots(1),$$

R being the radius of the larger section which is in contact. We have to find EG, the distance of the c. g. of the frustum from the base AB. Let α be the semi-vertical angle of the cone.

$\therefore \quad OE=R\cot\alpha,\ OF=r\cot\alpha \quad \therefore \quad h=EF=(R-r)\cot\alpha$ ......(2)

Volume of cone $OAB=\frac{1}{3}.\pi R^2.R\cot\alpha=\frac{1}{3}\pi R^3.\cot\alpha$;

Volume of cone $ODC=\frac{1}{3}\pi r^3\cot\alpha$; height of c. g. of cone OAB above $E=\frac{1}{4}.OE=\frac{1}{4}R\cot\alpha$; height of c. g. of cone ODC $=h+\frac{1}{4}r\cot\alpha$; and frustum $ABCD\equiv$ cone $OAB-$ cone ODC.

$$\therefore \quad EG=\frac{\frac{1}{12}.\pi R^4\cot^2\alpha-\frac{1}{3}\pi r^3\cot\alpha(h+\frac{1}{4}r\cot\alpha)}{\frac{1}{3}\pi\cot\alpha(R^3-r^3)}$$

$$=\frac{(R^4-4Rr^3+3r^4)\cot\alpha}{4(R^3-r^3)},\text{ substituting for } h \text{ from (2)}$$

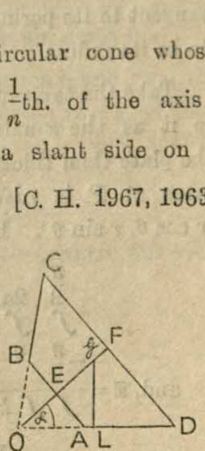
$$=\frac{(R-r)^2(R^2+2Rr+3r^2).h}{4(R-r)(R^3-r^3)}=\frac{h}{4}.\frac{R^2+2Rr+3r^2}{R^2+Rr+r^2}.$$

The frustum with tumble or slide,

according as $\tan\theta \lesseqgtr \tan\lambda=\mu$ *or*, $\frac{R}{EG}\lesseqgtr\mu$, [from (1)]

or, $\mu\gtrless\frac{4R}{h}.\frac{R^2+Rr+r^2}{R^2+2Rr+3r^2}$.

Ex. 2. A frustum of a uniform right circular cone whose semi-vertical angle is α is made by cutting off $\frac{1}{n}$th. of the axis; Prove that the frustum will rest with a slant side on a horizontal plane if $\tan^2\alpha<\frac{1}{4}.\frac{3n^4-4n^3+1}{n^3-1}$. [C. H. 1967, 1963]



Let OCD be the cone whose axis is OEF and from which the frustum $ABCD$ is made by cutting off the portion OAB, where $OE=\frac{1}{n}OF=\frac{1}{n}h$, writing $OF=h$. If G be the c. g. of the frustum, then the frustum will rest with side AD on the horizontal plane OAD, if the vertical through G viz GL meets the plane at a point L lying within AD *i.e.* if $OA<OL<OD$. As $OL<OD$, the condition is $OA<OL$. *or*. $OE\sec\alpha<OG\cos\alpha$.

or, $\frac{h}{n}\sec^2\alpha<OG$......(1).

Now, we have to find OG :

Frustum $ABCD \equiv$ cone $OCD -$ cone OBA.

Vol. of cone $OCD = \frac{\pi}{3} h^3 \tan^2\alpha$; vol. of cone $OBA = \frac{\pi}{3} \frac{h^3}{n^3} \tan^2\alpha$;

Distance of c. g. of cone OCD from $O = \frac{3}{4}h$, and that of cone $OAB = \frac{3}{4} \cdot \frac{h}{n}$.

$$\therefore \quad OG = \frac{\frac{\pi}{4} h^4 \tan^2\alpha - \frac{\pi}{4} \cdot \frac{h^4}{n^4} \tan^2\alpha}{\frac{\pi}{3} h^3 \tan^2\alpha \left(1 - \frac{1}{n^3}\right)} = \frac{3h}{4} \cdot \frac{n^4 - 1}{n(n^3 - 1)}.$$

$\therefore$ Condition (1) becomes, $\sec^2\alpha < \frac{3}{4} \cdot \frac{n^4 - 1}{(n^3 - 1)}$

or, $\tan^2\alpha < \frac{3}{4} \cdot \frac{n^4 - 1}{(n^3 - 1)} - 1 = \frac{3n^4 - 4n^3 + 1}{4(n^3 - 1)}$.

Hence the frustum will rest with a slant side on the horizontal plane if $\tan^2\alpha < \frac{1}{4} \cdot \frac{3n^4 - 4n^3 + 1}{n^3 - 1}$

Ex. 3. The thickness of a thin circular homogeneous plate at any point is proportional to the distance of the point from a tangent to its perimeter. Find the position of its c. g., taking a to be its radius. [C. H. 1956]

Take the tangent as the y-axis and the diameter perpendicular to it as the x-axis. If (r, θ) be the co-ordinates of a point P of the plate then thickness $t \propto r \cos\theta$. *or,* $t = kr \cos\theta$. Elementary mass at P is $rd\theta dr . kr . \cos\theta . \rho$, if $\rho =$ density, and its c. g. is $(r\cos\theta, r\sin\theta)$. From symmetry c. g. of the plate is on x-axis.

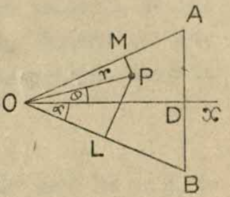
$$\text{and, } \bar{x} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2a\cos\theta} k\rho . r^3 \cos^2\theta dr d\theta}{\int \int k\rho . r^2 \cos\theta dr d\theta}.$$

$$= \frac{3}{4} \cdot (2a) \cdot \frac{\int_{0}^{\frac{\pi}{2}} \cos^6\theta d\theta}{\int_{0}^{\frac{\pi}{2}} \cos^4\theta d\theta}.$$

$$=\frac{3}{2}a.\frac{5.3.1}{6.4.2}.\frac{\pi}{2}.\frac{4.2.}{3.1.}.\frac{2}{\pi}=\frac{5}{4}a.$$

Hence the c. g. of the plate is on the diameter perpendicular to the tangent and at a distance $\frac{a}{4}$ from the centre.

Ex. 4. An isosceles triangular lamina is such that its mass per unit area at every point is proportional to the sum of the distances of the point from the equal sides of the triangle. Prove that the distance of the centre of gravity from the vertex is three-fourths of the altitude.



Let OAB be the lamina where $OA=OB$ and $\angle AOB=2\alpha$. Take O as origin and OD, perpendicular from O on AB as x-axis Then $\angle AOD=\angle BOD=\alpha$. From any point $P(r,\theta)$ of the lamina draw PL, PM $\perp^s$ on OB and OA respectively.

Then $PL=r\sin(\theta+\alpha)$ and

$$PM=r\sin(\alpha-\theta).$$

$\therefore\quad PL+PM=r[\sin(\theta+\alpha)+\sin(\alpha-\theta)]=2r\sin\alpha\cos\theta.$

$\therefore$ Element of mass at P is $rd\theta dr.2kr\sin\alpha\cos\theta$ and its c. g. is $(r\cos\theta, r\sin\theta)$. As OD is a line of symmetry, the c. g. of the lamina is on OD.

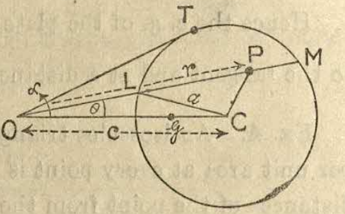
$$\text{And }\bar{x}=\frac{\int_{-\alpha}^{\alpha}\int_{0}^{h\sec\theta} rd\theta dr.2r^2\sin\alpha\cos\theta\cos\theta}{\int\int rd\theta dr.2r\sin\alpha\cos\theta},\text{ where } OD=h.$$

$$=\frac{3h}{4}.\frac{\int_{-\alpha}^{\alpha}\sec^4\theta\cos\theta\cos\theta\,d\theta}{\int_{-\alpha}^{\alpha}\sec^3\theta\cos\theta d\theta}=\tfrac{3}{4}h.$$

Hence the c. g. is on the altitude and at a distance of $\frac{3}{4}$th of the altitude from the vertex.

Ex. 5. The centre of inertia of a circular disc, the density of which varies inversely as the fourth power of the distance from an external point O in the plane, is the inverse point of O with respect to the boundary of the disc.

Let C be the centre of the disc and let $OC=c$. Take O as origin and OC as x-axis and let any point P of the disc be (r, θ). Then $\rho=\frac{k}{r^4}$ and the mass of the element at $P=\frac{k}{r^4}\cdot rd\theta dr$ and its c. g. is $(r\cos\theta,\ r\sin\theta)$. From symmetry, the c. g. is on OC.



$$\therefore\ \bar{x}=\frac{\int\limits_{-\alpha}^{\alpha}\int\limits_{r=OL}^{OM}\frac{k}{r^4}rd\theta dr.r\cos\theta}{\int\limits_{-\alpha}^{\alpha}\int\limits_{r=OL}^{OM}\frac{k}{r^4}rd\theta dr},$$

[The limits of θ are $-\alpha, \alpha$ where α is the angle between OC and the tangent OT drawn from O to the circle and $\therefore\ \sin\alpha=\frac{a}{c}$, where a = radius of circle. If the line OP cuts circle at L and M, then when θ is θ, r varies from OL to OM].

To find OL, OM in terms of θ, let $OL=t$.

From $\triangle OCL$, we have, $a^2=c^2+t^2-2ct\cos\theta$

or, $(t-c\cos\theta)^2=a^2-c^2\sin^2\theta$

$\therefore\ t=c\cos\theta\pm\sqrt{a^2-c^2\sin^2\theta}$

$\therefore\ OL=c\cos\theta-\sqrt{a^2-c^2\sin^2\theta}=t_1$,

$OM=c\cos\theta+\sqrt{a^2-c^2\sin^2\theta}=t_2$,

$$\therefore\ \bar{x}=\frac{\int\limits_{-\alpha}^{\alpha}\cos\theta\cdot\Big[-\frac{1}{r}\Big]_{t_1}^{t_2}d\theta}{\int\limits_{-\alpha}^{\alpha}\Big[-\frac{1}{2r^2}\Big]_{t_1}^{t_2}d\theta}$$

Now, $\frac{1}{t_1}-\frac{1}{t_2}=\frac{t_2-t_1}{t_1t_2}=\frac{2\sqrt{a^2-c^2\sin^2\theta}}{c^2-a^2}$

and $\frac{1}{t_1^2}-\frac{1}{t_2^2}=\frac{(t_2+t_1)(t_2-t_1)}{t_1^2t_2^2}=2c\cos\theta.2\sqrt{a^2-c^2\sin^2\theta}$

$$\therefore\quad \bar{x}=\left(\frac{c^2-a^2}{c}\right)\cdot\frac{\int_{-\alpha}^{\alpha}\cos\theta.\sqrt{a^2-c^2\sin^2\theta}.d\theta}{\int_{-\alpha}^{\alpha}\cos\theta\sqrt{a^2-c^2\sin^2\theta}.d\theta}=\frac{c^2-a^2}{c}$$

$\therefore$ If G be the c. g., we have $c.OG=c^2-a^2$

or, $OC^2-OC.OG=a^2$ *or*, $OC.(OC-OG)=a^2$

or, $CO.CG=a^2$.

$\therefore$ G is the inverse point of O with respect to the circle of centre C and radius a.

Ex. 6. The distance from the cusp of the centroid of the area of the cardioid $r=a(1+\cos\theta)$, when the density at any point varies as the nth power of the distance from the cusp is $\dfrac{(n+2)(2n+5)}{(n+3)(n+4)}\cdot a$.

The element of mass at $(r,\theta)=kr^n.rd\theta dr$ and its c. g. is $(r\cos\theta, r\sin\theta)$ From symmetry the c. g. of the cardioid is on its axis. If $\bar{x}$ be its distance from the origin which is the cusp, we have,

$$\bar{x}=\frac{\int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} kr^{n+2}.\cos\theta d\theta dr}{\int\int kr^{n+1}.d\theta dr}$$

$$=\frac{n+2}{n+3}\cdot a\cdot\frac{\int_{0}^{\pi}(1+\cos\theta)^{n+3}.\cos\theta d\theta}{\int_{0}^{\pi}(1+\cos\theta)^{n+2}d\theta}.$$

$$\int_{0}^{\pi}(1+\cos\theta)^{n+3}\cos\theta d\theta=2^{n+3}\int_{0}^{\pi}\left(\cos\frac{\theta}{2}\right)^{2n+6}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

To integrate, put $\frac{\theta}{2}=\phi$,

$$=2.2^{n+3}\int_{0}^{\frac{\pi}{2}}[(\cos\phi)^{2n+8}-(\cos\phi)^{2n+6}.(\sin\phi)^2]d\phi$$

$$=2^{n+4}\left[\tfrac{1}{2}\cdot\frac{\Gamma\left(\frac{2n+9}{2}\right)\Gamma(\frac{1}{2})}{\Gamma(n+5)}-\tfrac{1}{2}\cdot\frac{\Gamma\left(\frac{2n+7}{2}\right)\Gamma(\frac{3}{2})}{\Gamma(n+5)}\right]$$

$$=\frac{2^{n+3}\Gamma(\frac{1}{2})}{\Gamma(n+5)}\cdot\Gamma\left(\frac{2n+7}{2}\right)\left[\frac{2n+7}{2}-\frac{1}{2}\right]$$

$$=2^{n+3}.(n+3)\cdot\frac{\Gamma\left(\frac{2n+7}{2}\right)\Gamma(\frac{1}{2})}{\Gamma(n+5)}.$$

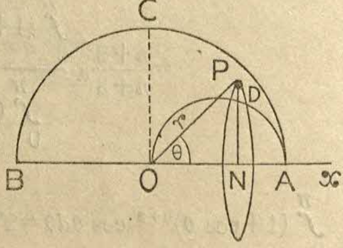
Similarly, $\int_0^{\pi}(1+\cos\theta)^{n+2}d\theta$

$$=2^{n+3}\int_0^{\frac{\pi}{2}}(\cos\phi)^{2n+4}d\phi=2^{n+2}\cdot\frac{\Gamma\left(\frac{2n+5}{2}\right)\Gamma(\frac{1}{2})}{\Gamma(n+3)}.$$

$$\therefore\quad \bar{x}=\frac{n+2}{n+3}a\cdot 2\frac{(n+3)(2n+5)}{2(n+4)(n+3)}=\frac{(n+2)(2n+5)}{(n+3)(n+4)}\cdot a.$$

Ex. 7. If the density at any point of a sphere of radius a directly varies as the distance from the centre and a sphere described on the radius as diameter be cut out, show that the centre of gravity of the remainder is at a distance $\frac{4}{63}a$ from the centre. [C. H. 1953]

Let BCA be a semi-circle of radius a whose centre is O and bounding diameter is BOA. On OA as diameter draw the semi-circle ODA. Let the whole figure be rotated about BOA through 2π; then the semi-circle BCA will describe a sphere of radius a and the semi-circle ODA will describe a sphere on radius OA as diameter. We have to find the c. g. of the larger sphere when the smaller sphere has been cut out. The solid which remains is generated by the revolution about BOA of the semi-circle BCA after the semi-circle ODA has been cut out from it. Take O as origin and OA as x-axis and let P (r, θ) by any point of the revolving area. Element of area at P is $rd\theta dr$. By a complete revolution it generates a ring of radius $PN=r\sin\theta$ and cross-section $rd\theta dr$, and it is homogeneous of density kr. ∴ mass



of elementary ring $=rd\theta dr.2\pi r\sin\theta kr$ and its c. g. is its centre N whose co-ordinates are $(r\cos\theta, 0)$ ∴ c. g. of given solid is on the x-axis and its distance from O is

$$\bar{x}=\frac{\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{a} r^4\sin\theta\cos\theta\,drd\theta+\int_{\frac{\pi}{2}}^{\pi}\int_0^{a} r^4\sin\theta\cos\theta\,drd\theta}{\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{a} r^3\sin\theta\,dr\,d\theta+\int_{\frac{\pi}{2}}^{\pi}\int_0^{a} r^3\sin\theta\,dr\,d\theta}$$

[When θ varies from 0 to $\frac{\pi}{2}$, r varies from $a\cos\theta$ to a as the area ODA has been cut out. And when θ varies from $\frac{\pi}{2}$ to π, r varies as usual, from 0 to a].

$$\text{Numerator}=\frac{a^5}{5}\int_0^{\frac{\pi}{2}}(1-\cos^5\theta)\sin\theta\cos\theta\,d\theta+\frac{a^5}{5}\int_{\frac{\pi}{2}}^{\pi}\sin\theta\cos\theta\,d\theta$$

$$=\frac{a^5}{5}\left[\left(\frac{1}{2}\sin^2\theta+\frac{\cos^7\theta}{7}\right)\Bigg|_0^{\frac{\pi}{2}}+\frac{1}{2}\sin^2\theta\Big)\Bigg|_{\frac{\pi}{2}}^{\pi}\right]$$

$$=\frac{a^5}{5}\left(-\frac{1}{7}\right)=-\frac{a^5}{35}.$$

$$\text{Denominator}=\frac{a^4}{4}\left[\int_0^{\frac{\pi}{2}}(1-\cos^4\theta)\sin\theta\,d\theta+\int_{\frac{\pi}{2}}^{\pi}\sin\theta\,d\theta\right]$$

$$=\frac{a^4}{4}\left[-\cos\theta\Bigg|_0^{\pi}+\frac{\cos^5\theta}{5}\Bigg|_0^{\frac{\pi}{2}}\right]$$

$$=\frac{a^4}{4}\left[2-\frac{1}{5}\right]=\frac{9a^4}{20}$$

$$\therefore\ \bar{x}=-\tfrac{4}{63}a.$$

Hence the c. g. of the given solid is on the line joining the centres of the two spheres and at a distance $\frac{4}{63}a$ from the centre of the larger one.

Another simpler solution is to break up the given solid into segments of concentric shells of homogeneous matter by spheres of centre O and radii varying from 0 to a. Consider such a shell bounded by radii r and $r+dr$. The segment of the shell, which is the element here, is bounded by two planes ⊥ to AB, one on the left of O at a distance r from O and the other to right of O at a distance $r\cos\theta$ from O. The area of the surface of the segment = the area of its projection on the circumscribing cylinder $=2\pi r(r+r\cos\theta)$

$$\therefore \text{ Mass of the element} = kr.\ 2\pi r(r+r\cos\theta).dr$$
$$=2\pi kr^3(1+\cos\theta)dr\ ;$$

and its c. g. is on OB at a distance from $O=\dfrac{r-r\cos\theta}{2}$, being mid way between the bounding planes.

∴ The c. g. of the solid is on B and its distance from O

$$=\frac{\int_0^a 2\pi kr^3(1+\cos\theta)dr.\frac{r}{2}(1-\cos\theta)}{\int_0^a 2\pi kr^3(1+\cos\theta)dr},\quad \text{where } \cos\theta=\frac{r}{a}$$

$$=\tfrac{1}{2}\frac{\int_0^a r^4\left(1-\frac{r^2}{a^2}\right)dr}{\int_0^a r^3\left(1+\frac{r}{a}\right)dr}=\tfrac{1}{2}\,a.\frac{\left(\frac{1}{5}-\frac{1}{7}\right)}{\left(\frac{1}{4}+\frac{1}{5}\right)}=\frac{}{63}a.$$

Or. We may first find the mass and c. g. of the smaller sphere and also the mass of the larger sphere—its c.g. is evidently O—and then find the c. g. of the compound body formed by the two spheres, the mass of the smaller one being considered negative.

Ex. 8. Find the c. g. of a portion of a solid sphere of radius a, contained in a right cone whose vertex is the centre of the sphere and whose semi-vertical angle is α, the density of the solid varying as the nth power of the distance from the centre.

Let O be the centre of the sphere and OA the axis of the cone. The given solid is generated by the revolution of a circular sector OAB about OA through 360°, where $OA=a$ and $\angle AOB=\alpha$, the density varying as the nth power of the distance from O. The mass of the elementary ring generated by the revolution about OA of the elementary area $rd\theta dr$ at a point $P(r, \theta)$ of the sector is $rd\theta dr.\ 2\pi r \sin\theta.\ kr^n$. and its c. g. is on OA at a distance $r\cos\theta$ from O.

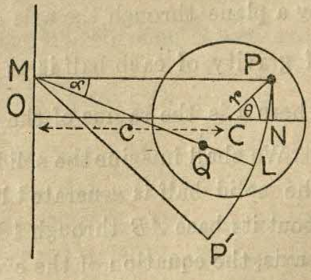
$\therefore$ The c. g. of the solid is on OA and its distance from O

$$=\frac{\int_0^{\alpha}\int_0^{a} 2\pi k.r^{n+2}.\ \sin\theta d\theta dr.\ r\cos\theta}{\int_0^{\alpha}\int_0^{a} 2\pi k.r^{n+2}\sin\theta d\theta dr}$$

$$=\frac{n+3}{n+4}.a.\ \frac{\frac{1}{2}(1-\cos^2\alpha)}{(1-\cos\alpha)}=\frac{n+3}{n+4}.\frac{a}{2}.\ (1+\cos\alpha).$$

Ex. 9. A portion of an anchor ring is formed by the revolution of a circular area of radius a about a line in its plane at a distance c from its centre, where $c>a$. If 2α be the angle through which it revolves, prove that the centre of gravity of the solid is at a distance from the line equal to $\frac{4c^2+a^2}{4c}.\ \frac{\sin\alpha}{\alpha}$.

[C. H. 1957]



Let C be the centre of the circular area and MO the line about which the area revolves. OC, the perpendicular from C on MO, is c. Take origin at C and OC as x-axis and let $P(r, \theta)$ be any point of the area. Its area is $rd\theta dr$ and due to the revolution about MO through an angle 2α, it generates a portion of a ring, PP', whose centre is M, the foot of the perpendicular from P on MO, whose radius is $PM=c+r\cos\theta$, whose length is $2\alpha.\,PM$ and whose cross-section is $rd\theta dr$. Therefore the volume of this element is $rd\theta dr.\ 2\alpha.(c+r\cos\theta)$. Its c. g. is on ML, which bisects the angle $PMP'=2\alpha$, and its distance from M is $PM.\dfrac{\sin\alpha}{\alpha}=(c+r\cos\theta).\dfrac{\sin\alpha}{\alpha}\cdot$ The c. g.'s of all such elementary volumes are in the plane through MO and ML.

$\therefore$ The distance of the c. g. of the anchor ring from the line MO

$$=\frac{\int_0^{2\pi}\int_0^a rd\theta dr.\ 2\alpha.(c+r\cos\theta)^2.\dfrac{\sin\alpha}{\alpha}}{\int_0^{2\pi}\int_0^a rd\theta dr.\ 2\alpha.(c+r\cos\theta)}$$

$$=\frac{\sin\alpha}{\alpha}.\frac{\int_0^{2\pi}\left(c^2\dfrac{a^2}{2}+2c\cos\theta.\dfrac{a^3}{3}+\cos^2\theta.\dfrac{a^4}{4}\right)d\theta}{\int_0^{2\pi}\left(c.\dfrac{a^2}{2}+\cos\theta.\dfrac{a^3}{3}\right)d\theta}$$

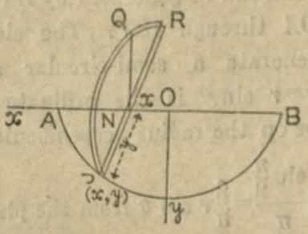
$$=\frac{\sin\alpha}{\alpha}.\frac{\left[\pi a^2c^2+\dfrac{a^4}{8}\ 2\pi\right]}{\pi.a^2c}=\frac{a^2+4c^2}{4c}.\frac{\sin\alpha}{\alpha}.$$

Ex. 10. A uniform solid is bounded by the surface formed by the revolution of a cycloid about its base and is then cut in halves by a plane through the axis of revolution. Show that the centre of gravity of each half is at a distance $\dfrac{7a}{3\pi}$ from the plane face, where a is the radius of the generating circle of the cycloid.

We shall imagine the solid to be cut by the plane of the paper. The solid half is generated by the revolution of the cycloid APB about its base AB through 180°. Taking O as origin and OA as x-axis, the equation of the cycloid is

$$\left.\begin{aligned}x&=a(\theta+\sin\theta)\\ y&=a(1+\cos\theta)\end{aligned}\right\}\ldots\ldots(1)$$

Let P be any point on the arc of the cycloid and (x, y) its co-ordinates. Then its ordinate PN is y and ON is x. Consider the strip PN of thickness dx. By revolving about AB through 180° it generates the semi-circular disc PQR whose centre is N, radius is $PN=y$ and whose thickness is dx. Therefore its volume is $\frac{1}{2}\pi y^2 dx$ and its c. g. is on NQ, the radius perpendicular on the diameter PNR at a distance $\frac{4y}{3\pi}$ from N. We divide the solid into such elementary semi-circular discs, and the c.g. of the solid is, from symmetry, on the line through O perpendicular to the plane of the paper. If G be the c. g. of the solid we have,



$$OG=\frac{\int \frac{1}{2}\pi y^2 dx \cdot \frac{4y}{3\pi}}{\int \frac{1}{2}\pi y^2 dx}=\frac{4a}{3\pi}\cdot\frac{\int_{-\pi}^{\pi}(1+\cos\theta)^4 d\theta}{\int_{-\pi}^{\pi}(1+\cos\theta)^3 d\theta}$$

$$=\frac{4a.2}{3\pi}\cdot\frac{\int_0^{\pi}\cos^8\frac{\theta}{2}d\theta}{\int_0^{\pi}\cos^6\frac{\theta}{2}d\theta}=\frac{8a}{3\pi}\cdot\frac{\int_0^{\frac{\pi}{2}}\cos^8\phi d\phi}{\int_0^{\frac{\pi}{2}}\cos^6\phi d\phi}$$

$$=\frac{8a}{3\pi}\cdot\frac{7.5.3.1}{8.6.4.2}\cdot\frac{\pi}{2}\cdot\frac{6.4.2}{5.3.1}\cdot\frac{2}{\pi}=\frac{7a}{3\pi}.$$

Hence the c. g. of the solid is on the line through O perpendicular to the bisecting plane at a distance $\frac{7a}{3\pi}$ from O.

Ex. 11. A solid is bounded by half the surface formed by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis, and by a plane base through its axis ; show that the distance of its centre of gravity from the axis is $\frac{63}{128}a$ and its distance from the pole, measured parallel to the axis is $\frac{4a}{5}$.

If OA be the axis of the generating cardioid, then the given solid can be generated by revolving one-half of the cardioid about OA through 180°. The elementary area $rd\theta dr$ at $P(r, \theta)$ will generate a semi-circular *ring* of mass $\rho.\pi y.rd\theta dr$, where $y=r \sin \theta$ is the ordinate of P and the c. g. of the ring will be on the radius perpendicular to the plane base at a distance $y\dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}=\dfrac{2}{\pi}\cdot r \sin \theta$ from the plane. Hence the co-ordinates of the c. g. of the ring, in the plane through OA perpendicular to the base and referred to O as origin and OA as x-axis are $x=r \cos \theta$, $y=\dfrac{2}{\pi} r \sin \theta$.

$$\therefore \quad \bar{x}=\frac{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r \cos\theta\, \pi\rho.r^2 \sin\theta d\theta dr}{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} \pi\rho.r^2 \sin\theta d\theta dr}=\frac{4a}{5},$$

$$\text{and} \quad \bar{y}=\frac{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} \frac{2}{\pi}\cdot r \sin\theta.r^2 \sin\theta d\theta dr}{\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta d\theta dr}=\frac{63}{128}a,$$

by evaluating the integrals.

Hence the result.

Ex. 12. A curve is such that if G be the c. g. of any arc AB of it then OG bisects the angle AOB, where O is a fixed point. Show that the curve is either a lemniscate or a circle.

It is the converse of Ex. 2 of Art. 5·9.

We take O as origin and OA as initial line and consider B to be a variable point on the curve, Let $\angle AOB=\phi$. We have to find the co-ordinates of G, the c. g. of arc AB. Let the co-ordinates be $\bar{x}$, $\bar{y}$. We divide the arc AB into elementary arcs; let PQ be such an element where P is (r, θ) and arc $AP=s$.

$\therefore$ mass of the element is $dm=\rho ds$ and its c. g. is $(r \cos \theta, r \sin \theta)$.

$$\therefore \quad \bar{x}=\frac{\int_{\theta=0}^{\theta=\phi} \rho ds.r\cos\theta}{\int \rho ds},\quad \bar{y}=\frac{\int_{\theta=0}^{\theta=\phi} \rho ds.r\sin\theta}{\int \rho ds}.$$

We have, $\tan AOG=\frac{\bar{y}}{\bar{x}}$. But $\angle AOG=\frac{1}{2}\angle AOB=\frac{\phi}{2}$.

$$\therefore \quad \tan\frac{\phi}{2}=\frac{\int_{\theta=0}^{\theta=\phi} r\sin\theta ds}{\int_{\theta=0}^{\theta=\phi} r\cos\theta ds},\text{ for all }\phi.$$

Let $r=f(\theta)$ be the equation of the curve.

Then $\frac{ds}{d\theta}=\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}=$ a function of θ.

Hence we may write $rds=F(\theta)d\theta \quad \ldots \quad (1)$

$\therefore$ we have $\tan\frac{\phi}{2}.\int_0^\phi F(\theta)\cos\theta d\theta=\int_0^\phi F(\theta)\sin\theta d\theta.$

or, differentiating both sides with respect to ϕ,

$$\tan\frac{\phi}{2}.F(\phi)\cos\phi+\tfrac{1}{2}\sec^2\frac{\phi}{2}\int_0^\phi F(\theta)\cos\theta d\theta=F(\phi)\sin\phi.$$

$\therefore \quad \int_0^\phi F(\theta)\cos\theta d\theta=F(\phi).\sin\phi$, after a little simplification.

Differentiating again with respect to ϕ, we have,

$$F(\phi)\cos\phi=F(\phi)\cos\phi+F'(\phi)\sin\phi \quad \textit{or,} \quad F'(\phi)\sin\phi=0.$$

As this holds for all values of ϕ, we have $F'(\phi)=0$

or, $F(\phi)=$ constant. $\therefore$ $F(\theta)=$ constant $=a^2$, say.

$$\therefore \quad r\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}=a^2 \quad [\text{ from (1) }]$$

Evidently $r=a$ is a solution. Simplifying the differential equation we have

$$d\theta=\frac{\pm rdr}{\sqrt{a^4-r^4}} \quad \therefore \quad \theta+\alpha=\int\frac{\pm rdr}{\sqrt{a^4-r^4}}$$ where α is a constant; putting $r^2=a^2\cos t$, and integrating, we have

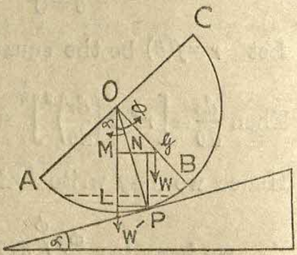
$$2(\theta+\alpha)=\pm t \quad \therefore \quad r^2=a^2\cos 2(\theta+\alpha).$$

If we take Ox as the initial line where $\angle AOx = -\alpha$, the equation becomes $r^2 = a^2 \cos 2\theta$. Thus the curve is either $r = a$, a circle or $r^2 = a^2 \cos 2\theta$, a lemniscate.

Ex. 13. A thin hemispherical bowl of weight W contains a weight W' of water and rests on a rough inclined plane of inclination α. Show that the plane of the top of the bowl makes an angle ϕ with the horizontal given by $W \sin \phi = 2(W + W') \sin \alpha$.

[C. H. 1955]

The bowl ABC rests in equilibrium on the inclined plane touching the latter at P. The weight W' of the liquid acts in the vertical line passing through O, the centre of the bowl and the weight W of the bowl acts vertically at its c g. G, which is the middle point of the radius OB perpendicular to the plane face AOC. Forces which keep the bowl in equilibrium are the two weights W, W' and the resultant reaction at P. $\therefore$ algebraic sum of the moments of W and W' about P is zero. From P and G draw PL and GM perpendiculars on the vertical through O.



Draw PN perpendicular on GM. Then moment of W' about $P = W'.PL = W'.a \sin \alpha$, anticlockwise, and moment of W about $P = W.GN = W(GM - PL) = W(OG \sin \phi - a \sin \alpha)$

$$= W\left(\frac{a}{2} \sin \phi - a \sin \alpha\right), \text{ clockwise.}$$

[$\angle MOG$ = inclination of face AOC to horizontal $= \phi$].

$$\therefore \quad W'a \sin \alpha = aW(\tfrac{1}{2} \sin \phi - \sin \alpha)$$

$$or, \quad W \sin \phi = 2 \sin \alpha . (W + W').$$

Miscellaneous Examples

1. Find the c. g. of the area of a cardioid, when the distance varies as the cube of the distance from the origin.

2. In a semi-circular disc, bounded by a diameter OA, the density at any point varies as the distance from O. Find the c. g.

3. The locus of the c. g. of all equal segments cut off from a parabola is an equal parabola.

4. A chord PQ of an ellipse, centre C, always passes through a fixed point O. Prove that the locus of the c. g. of the triangle CPQ is a similar ellipse.

5. Show that the circle is the only curve in which the centroid of the area included between the curve and two radii drawn from a fixed point always lies on the bisector of the angle between the radii.

6. An isosceles triangular lamina is such that its mass per unit area at every point varies as the sum of its distances from the equal sides of the triangle. Prove that the distance of the c. g from the vertex is three-fourth of the altitude.

7. A sphere of radius a is inscribed in a cone of vertical angle 2α. Show that the distance from the centre of the sphere of the c. g. of that part of the volume of the cone which lies between its vertex and the nearer part of the surface of the sphere is $\frac{a}{4}\cdot\frac{(1+\sin\alpha)^2}{4\sin\alpha}$.

8. A circular cylinder is terminated obliquely by a plane. Show that if L and l are the longest and the shortest generating lines on the cylinder, and r the radius of the circular section, the c. g. of the cylinder lies at the mid-point of a line which is parallel to the axis and at a distance $\frac{(L-l)}{(L+l)}\cdot\frac{r}{4}$ from it. [C. H. 1950]

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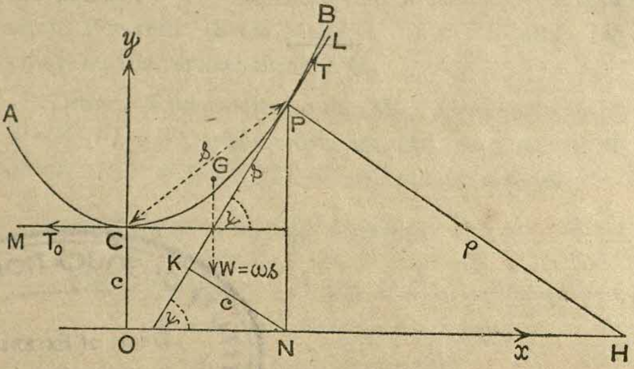
CHAPTER VI

EQUILIBRIUM OF FLEXIBLE STRINGS

6·1. In this Chapter we shall discuss the equilibrium of *perfectly flexible inextensible* strings under the action of given forces. The cross-section of the string is assumed to be so small that the string may be regarded as a 'line of matter.' The string is called perfectly flexible when it offers no resistance to its bending at any point; and this assumes that the internal forces exerted by one portion of the string on the other across the normal section at any point of the string reduce to a single force whose line of action is along the tangent to the string at the point. This single force is called the tension of the string at the point and is represented by T.

6·2. *A heavy uniform inextensible string is suspended from two given points and is in equilibrium in a vertical plane under the action of gravity only. To find the equation of the curve in which it hangs.*



By "uniform string" we mean that the string has the same cross-section throughout its length and has the same density at every point.

Let the string be suspended from the points A, B and let it assume the form ACB when in equilibrium, the string lying in a vertical plane (which is assumed to be the plane of the paper). Let C be the lowest point of the string; then the tangent to the string at C is horizontal. Let the vertical through C be taken as the axis of y, the point O on it as the origin and the horizontal through O in the plane of ACB as the axis of x. Let $P(x, y)$ be any point of the string and consider the separate equilibrium of the finite portion CP of the string. The portion CP is in equilibrium under the action of the three following forces :

(i) the tension T at P acting along the tangent PL at P, which makes an angle ψ with Ox (it is the resultant action of the portion PB on CP) ;

(ii) the tension T_0 at C acting along the horizontal tangent CM at C (it is the resultant action of the portion CA on CP) ;

(iii) the weight W of the portion CP acting vertically downwards through its c.g. G.

Since these three forces must meet in a point, the c.g. G must lie on the vertical through the point where the forces T and T_0 meet.

Resolving the forces horizontally and vertically, we get,

$$T \cos \psi = T_0, \ldots\ldots T \sin \psi = W \ldots\ldots\ldots (1)$$

If ω=wt. per unit length of the string and

s=length of the arc CP,

we have $W = \omega s$. Also let $T_0 = c\omega$.

From (1), we have, $\tan \psi = \frac{W}{T_0} = \frac{s}{c}$ *or*, $\mathbf{s = c \tan \psi} \ldots\ldots (2)$

which is the *intrinsic equation of the curve ACB.*

We shall now derive its cartesian equation :

we have $\frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi} = \cos \psi .\, c \sec^2 \psi$ [From (2)]

$= c \sec \psi$.

Integrating, $x + A = c \log (\sec \psi + \tan \psi)$

At C, $x = 0, \psi = 0$ $\therefore$ $A = 0$

$$\therefore \quad \mathbf{x = c \log (\sec \psi + \tan \psi)} \ldots\ldots (3)$$

Again, $\frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi} = \sin \psi .\, c \sec^2 \psi = c \sec \psi \tan \psi$,

Integrating, $y + B = c \sec \psi$.

Let the origin be taken at a distance c below the lowest point C of the string. Then $OC = c$, and we have at C, $y = c, \psi = 0$ $\therefore$ $B = 0$.

$$\therefore \quad \mathbf{y = c \sec \psi} \ldots\ldots\ldots (4)$$

From (3), $\sec\psi+\tan\psi=e^{\frac{x}{c}}$

$$\therefore\ \sec\psi-\tan\psi=\frac{1}{\sec\psi+\tan\psi}=e^{-\frac{x}{c}}.$$

Adding and subtracting, we get, $\sec\psi=\cosh\frac{x}{c}$

and $\tan\psi=\sinh\frac{x}{c}$.

Hence, $\mathbf{y=c\cosh\frac{x}{c}}$ [From (4)].........(5)

and $\mathbf{s=c\sinh\frac{x}{c}}$. [From (2)].........(6)

Equation (5) is the cartesian equation of the curve ACB in which the heavy string hangs freely. This curve is known as the *Common Catenary*. The lowest point C is called its *vertex*, the axis Oy is called its *axis* and the axis Ox which is at a distance c below the vertex is called the *Directrix* of the Catenary.

The constant c is called the *parameter* of the Catenary and since it is the only constant appearing in its Cartesian equation (5), it follows that Catenaries having the same parameter are equal.

Eqn. (5) does not change when $-x$ is written for x, so the Catenary is symmetrical about its axis. $\cosh\frac{x}{c}>0$ for all x and $\to\infty$ as $x\to\infty$. ∴ The Catenary lies wholly above its directrix and extends to infinity upwards on both sides of the axis.

6·3. Eqns. (1) of Art 6·2 give two important properties of the Catenary :

(1) The horizontal tension is constant, being the same at every point and is equal to wc.

(2) The vertical tension at any point P is equal to the wt. of CP which is ws.

The formulæ (2), (3), (4), (5), (6) are very important. We shall now deduce some other important ones.

From (2) and (4), we get $\mathbf{y}^2=c^2\sec^2\psi=c^2+c^2\tan^2\psi$

$=\mathbf{c^2+s^2}$......(7)

From (5) and (6), adding, we get $\mathbf{s+y=ce^{\frac{x}{c}}}$......(8)

From (1) and (4), we get $\mathbf{T}=T_0\sec\psi=wc\sec\psi=\mathbf{wy}$. ....(9)

This last relation shows that the tension at any point of the string is equal to the wt. of a portion of the string whose length is equal to the height of the point above the directrix. Therefore, if a uniform heavy string hangs freely in equilibrium after passing over smooth pegs, the **catenaries** formed by the various portions of the string **have the same directrix** and the **free ends** of the string, if any, will also be **on the same directrix.**

Finally, differentiating (2), we have $\rho=\dfrac{ds}{d\psi}=\mathbf{c}\ \mathbf{sec}^2\psi$......(10)

$$or,\quad \rho=\mathbf{y}\ \mathbf{sec}\,\psi\ .....(11)$$

At C, $\psi=0$. $\therefore$ ρ at $C=c$. So c is large or small according as the curve is flat or much curved near its vertex. *Hence if a string is tightly stretched between two points A, B in the same horizontal line then C is very large compared to AB.*

6·4. We shall now state some geometrical properties of the Catenary : In fig. of Art. 6·2, $PN=y$ is the ordinate of P,

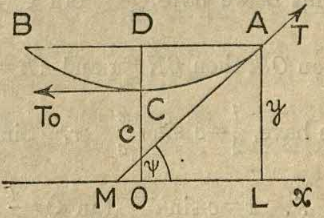
NK is the perpendicular from N on the tangent PL at P and PH is the normal at P meeting the directrix at H. $\angle NPH=\angle PNK=\psi$.

$$\therefore\quad \left.\begin{array}{l}\mathbf{NK}=NP\cos\psi=y\cos\psi=c\sec\psi.\cos\psi=\mathbf{c}\\ \mathbf{PK}=NP\sin\psi=y\sin\psi=c\sec\psi\sin\psi=c\tan\psi=\mathbf{s}.\end{array}\right\}(12)$$

and $\mathbf{PH}=PN\sec\psi=y\sec\psi=\rho$ from (11).

Examples Worked

Ex. 1. A string of length $2l$ feet is suspended between two points at the same level, and the lowest point of the string is d feet below the points of suspension. If the weight per unit length of the string be ω lbs, find the horizontal component of the tension. [C. H. 1963]

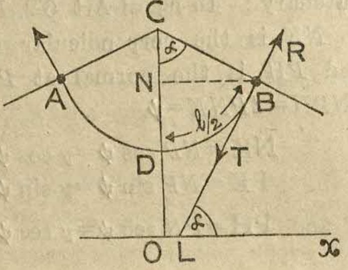


Let ACB be the string ; A, B are the points of suspension in same level and C is the lowest point of the string. Ox is the directrix of the common catenary in which the string hangs *i.e.* ACB and OCD is the vertical through C, the lowest point. We take Ox, OC as axes of x and y ; we have $DC=d$, arc $CA=l$, $CO=c$, the parameter of the catenary. If $y=AL$ be the ordinate of A, we have, $y=DO=d+c$. Using the formula

$y^2=c^2+s^2$ for the point A, we have, $(d+c)^2=c^2+l^2$ or, $c=\frac{l^2-d^2}{2d}$; this gives the parameter of the catenary. Now, horizontal component of the tension at any point of the string is constant and equal to $c\omega$. $\therefore$ Horizontal component $=\omega.\frac{l^2-d^2}{2d}$.

Ex. 2. Two smooth rods are situated in the same vertical plane and make equal angles α with the downward vertical. A uniform chain of length l hangs by weightless rings from the rods. Prove that the distance between the rings is $l \cot \alpha \sinh^{-1}(\tan \alpha)$.

Let CA, CB be the rods and ADB the chain when in equilibrium, A, B being the rings at its ends. From symmetry, D, the lowest point of the catenary ADB will be on the vertical CO through O; Ox is the directrix of the catenary. We have,



$$\text{arc } DB=\frac{l}{2},\ \angle OCB=\alpha.$$

Let BL be the tangent at B to the catenary. Consider the equilibrium of the ring B; only two forces are acting on it, the normal reaction R of rod $CB \perp$ to the latter at B, and the tension T of the chain acting along the tangent BL. $\therefore$ *R and T must act along the same straight line.* Thus BL is $\perp$ to CB. $\therefore$ $\angle BLx=\angle BCO=\alpha$; using the formula $s=c \tan \psi$ for the point B, we have, $\frac{l}{2}=c \tan \alpha$ or, $c=\frac{l}{2}\cot\alpha$. Let BN be $\perp$ from B on OC, then $BN=x$ and $AB=2BN$; from the formula $s=c \sinh\frac{x}{c}$ we have, $\frac{l}{2}=c \sinh\frac{x}{c}$ or, $\operatorname{Sinh}\frac{x}{c}=\frac{l}{2}\cdot\frac{2}{l}\tan\alpha=\tan\alpha$.

$$\therefore\ x=c\sinh^{-1}(\tan\alpha)=\frac{l}{2}\cot\alpha\sinh^{-1}(\tan\alpha).$$

$$\therefore\ \text{Distance between the rings}=AB$$
$$=2x=l\cot\alpha\sinh^{-1}(\tan\alpha).$$

Ex. 3. A uniform chain, of length l, is to be suspended from two points, A and B, in the same horizontrl line so that either terminal tension is n times that at the lowest point. Show that the span AB must be $\dfrac{l}{\sqrt{n^2-1}}\log_e[n+\sqrt{n^2-1}]$. [C. H. 1960]

Let C be the lowest point and c the parameter of the catenary in which the chain hangs. Let tension T at A make an angle ψ with the directrix. (See fig. Ex. 1, p. 205)

Then $T\cos\psi=T_0$, T_0 being tension at C.

Also given $T=nT_0$

$\therefore\ n\cos\psi=1$ *or*, $\sec\psi=n$......(1).

Using formula $s=c\tan\psi$ for the point A, since $CA=\frac{1}{2}l$,

$$\text{we get } \frac{l}{2}=c\tan\psi=c\sqrt{n^2-1}\quad[\text{ from } (1)].$$

$$or,\quad c=\frac{l}{2\sqrt{n^2-1}}\text{......}(2)$$

Finally using formula $x=c\log(\sec\psi+\tan\psi)$ for the point A,

$$\text{we get,}\quad \frac{AB}{2}=c\log(\sec\psi+\tan\psi)$$

$$or,\quad AB=\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1}),\quad[\text{ from (1) and (2) }]$$

Ex. 4. If α and β be the angles which a string of length l makes with the vertical at the points of support, show that the height of one point above the other is $\dfrac{l\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$.

[C. H. 1965]

Let AC be the string, A, C the points of support, and let D be the vertex of the catenary formed by AC (See fig. Ex. 7 below). Let Ox be the directrix, then $DO=c$, the parameter of the catenary. Let arc $DC=s_0$, then arc $DA=s_0+l$. Using the formula, $s=c\tan\psi$ for the points A and C, we have, $s_0+l=c\tan(90-\alpha)=c\cot\alpha$ and $s_0=c\tan(90-\beta)=c\cot\beta$, α, β being the inclinations to the vertical of the tangents at A and C respectively. $\therefore\ l=c(\cot\alpha-\cot\beta)$......(1) Again, let $k=EC$ be the height of A above C; then if $AM=y$ be the height of A above Ox, that of C is $CN=(y-k)$. Using the formula y

$=c\sec\psi$ for the points A and C, we have $y=c\sec(90-\alpha)$ $=\operatorname{cosec}\alpha$, and $y-k=c\sec(90-\beta)=c\operatorname{cosec}\beta$.

$$\therefore\quad k=c(\operatorname{cosec}\alpha-\operatorname{cosec}\beta)\ldots\ldots(2)$$

From (1) and (2) we have, $\dfrac{k}{l}=\dfrac{\operatorname{cosec}\alpha-\operatorname{cosec}\beta}{\cot\alpha-\cot\beta}$

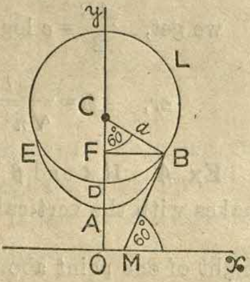
$$=\frac{\sin\beta-\sin\alpha}{\sin(\beta-\alpha)}=\frac{\cos\dfrac{\beta+\alpha}{2}}{\cos\dfrac{\beta-\alpha}{2}}.$$

Hence $k=\dfrac{l.\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$.

Ex. 5. Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}. \qquad \text{[C. H. 1950]}$$

Let A be the lowest point of the chain $ABLEA$ of which the portion BLE is in contact with the pulley of centre C and the portion EAB hangs in the form of a common catenary, A being its vertex and BE being horizontal from symmetry. Let vertical CA meet the pulley at D and the directrix Ox at O. Length of portion BLE is $\frac{2}{3}\times 2\pi a$. $\therefore$ Length of arc EDB of the pulley not in contact with the string $=\frac{1}{3}.2\pi a$.



$\therefore$ angle $ECB=\dfrac{2\pi}{3}$. $\therefore$ $\angle DCB=\dfrac{\pi}{3}$. The tangent at B to the catenary is the tangent at B to the pulley. So if the tangent at B meets Ox at M, we have $\angle BMx=\angle BCD=\dfrac{\pi}{3}=60°$. Draw $BF\perp$ on OC $\therefore$ $BF=BC\sin 60°=\dfrac{a\sqrt{3}}{2}$

For the point B, $x=BF=\dfrac{a\sqrt{3}}{2}$ and $\psi=60°$.

$\therefore$ using the formula, $x=a\log(\sec\psi+\tan\psi)$, we get $\frac{a\sqrt{3}}{2}=c\log(2+\sqrt{3})$ *or*, $c=\frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$, the parameter of the catenary.

Now length of arc AB of the catenary $=c\tan\psi=c\tan 60°$

$$=\sqrt{3}.c=\frac{3a}{2\log(2+\sqrt{3})}.$$

$\therefore$ Length of EAB, the hanging portion of the chain

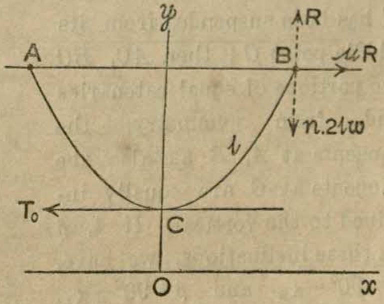
$$=2\text{ arc }AB=\frac{3a}{\log(2+\sqrt{3})};$$

and length of the portion in contact with the pulley $=\frac{4a\pi}{3}$.

$\therefore$ Total length of the chain $=a\left[\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right]$.

Ex. 6. A heavy chain of length $2l$ has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log[\lambda+\sqrt{1+\lambda^2}]$,

where $\frac{1}{\lambda}=\mu(2n+1)$, and μ is the coefficient of friction.



Let AB be the greatest possible distance of the ring B from A so that the equilibrium of the system, the chain and the ring, may be maintained. Let C be the vertex of the catenary; then $CB=l$. Let c be the parameter of the catenary and ω the weight of unit length of the string.

Then weight of ring $=n\cdot 2l\omega$. Tension at C, $T_0=c\omega$.

Since AB is the greatest possible distance, the equilibrium is limiting. Let R be the normal reaction of the rod on the ring; then μR is the force of friction.

Considering the equilibrium of the system formed by the portion CB of the chain and the ring, we get

$$\mu R=T_0=c\omega\ldots\ldots(1)\qquad \omega l+n.2l\omega=R\ldots\ldots(2)$$

From (1) and (2), we get $c=\mu l(2n+1)=\dfrac{l}{\lambda}\cdots\cdots(3)$

Again, using the formula $s=c\tan\psi$ for the point B of the chain, we get $l=c\tan\psi$, where ψ=inclination of tangent at B to directrix.

$$or,\quad \tan\psi=\frac{l}{c}=\lambda \quad [\text{ from } (3)\ ]\cdots\cdots(4)$$

Finally, using the formula, $x=c\log(\sec\psi+\tan\psi)$ for the point B,

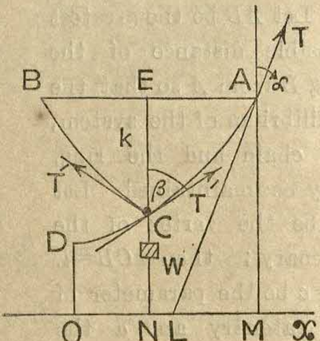
$$\text{we get } \frac{AB}{2}=\frac{l}{\lambda}\log\left(\sqrt{1+\lambda^2}+\lambda\right) \quad [\text{from (3) and (4)}]$$

$$or,\quad AB=\frac{2l}{\lambda}\log\left(\lambda+\sqrt{1+\lambda^2}\right).$$

Ex. 7. A uniform inextensible chain of weight C is suspended by its extremities from two points on the same level and a weight W is attached to its lowest point. If α_1 and α_2 are the angles with the horizon made by the tangents to the curve at W and the points of suspension respectively, show that

$$C=W\left[\frac{\tan\alpha_2}{\tan\alpha_1}-1\right]. \qquad [\text{C. H } 1961]$$

Let A, B be the points of support and let ACB be the chain in equilibrium when a weight W has been suspended from its middle point C; then AC, BC are portions of equal catenaries and from symmetry, the tangents at A, B as also the tangents at C are equally inclined to the vertical. If α, β be these inclinations, we have, $\alpha=90^\circ-\alpha_2$ and $\beta=90^\circ-\alpha_1$. Let T, T' be the tensions at A and C. First consider the equilibrium of the whole system, the chain and the weight W. Forces acting on it are the terminal tensions T, T at A and B, the weight C of the chain and the weight W.



$\therefore$ Resolving vertically, we have, $2T\cos\alpha=W+C\cdots\cdots(1)$

Next from equilibrium of weight W, resolving vertically, we have $2T'\cos\beta=W\cdots\cdots(2)$

Also, as the horizontal component of the tension is constant, we have, $T \sin \alpha = T' \sin \beta \cdots\cdots(3)$

From (1) and (2), we have,

$$\frac{W+C}{W} = \frac{T \cos \alpha}{T' \cos \beta} = \frac{\sin \beta}{\sin \alpha} \cdot \frac{\cos \alpha}{\cos \beta} \quad [\text{ from } (3)\ ]$$

$$= \frac{\tan \beta}{\tan \alpha} = \frac{\cot \alpha_1}{\cot \alpha_2} = \frac{\tan \alpha_2}{\tan \alpha_1},$$

$$[\text{ as } \alpha = 90^\circ - \alpha_2,\ \beta = 90^\circ - \alpha_1\ ]$$

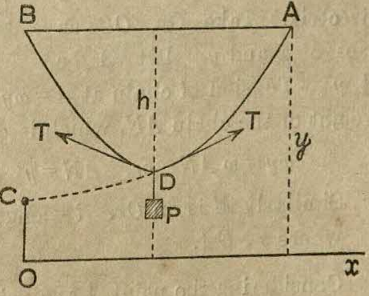
$$\text{or,} \quad C = W\left[\frac{\tan \alpha_2}{\tan \alpha_1} - 1\right]$$

If $C = nW$, we have, $(n+1) = \dfrac{\tan \alpha_2}{\tan \alpha_1}$

or, $\tan \alpha_2 = (n+1) \tan \alpha_1$.

Ex. 8. A uniform chain, of length $2l$, and weight W, is suspended from two points, A and B, in the same horizontal line. A load P is now suspended from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is $\frac{1}{2}\left[P\frac{l}{h} + W \cdot \frac{h^2+l^2}{2hl}\right]$. [C. H. 1949]

From symmetry, the two equal portions AD and BD of the chain must be arcs of equal catenaries; the tensions at D of the two portions must be equal and equally inclined to the vertical. Let each tension be T inclined at an angle ψ to the horizontal. Let AD be produced to C, the vertex of the catenary of which AD is a part and let c be the parameter of the catenary and Ox the directrix, OC being the y-axis.



From equilibrium of wt. P, we have, $2T \sin \psi = P$.

But we know $T \sin \psi = \omega s$, where $CD = s$, and $\omega = \dfrac{W}{2l}$.

$$\therefore \quad 2\omega s = P \quad \text{or,} \quad s = \frac{P}{2\omega} = \frac{Pl}{W} \cdots(1)$$

If y = height of A above Ox, that of $D=y-h$.
Using the formula $y^2=c^2+s^2$ for the points A and D,

we get $y^2=c^2+(s+l)^2$, where $CA=CD+DA=s+l$.

and $(y-h)^2=c^2+s^2$.

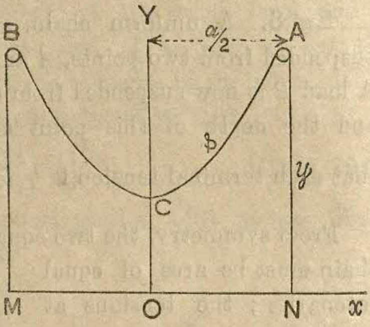
Subtracting, we get, $2hy-h^2=2sl+l^2$.

$$or,\quad y=\frac{2sl+l^2+h^2}{2h}.$$

$$\therefore\ \text{Tension at } A=\omega y=\frac{W}{2l}\left[\frac{l}{h}\cdot P\frac{l}{W}+\frac{l^2+h^2}{2h}\right]$$

$$=\frac{1}{2}\left[P\frac{l}{h}+W\cdot\frac{l^2+h^2}{2hl}\right].$$

Ex. 9. A heavy uniform chain hangs in equilibrium over two smooth pegs A, B on the same level and at a distance a apart with the two ends hanging freely and the central part in the form of a curve. If the chain is of length l, show that $l \geqslant ae$. [C. H. 1961]



Let $MBCAN$ be the chain, C being the vertex of the catenary BCA of parameter c say, and let Ox be its directrix. Take Ox, OC as axes of x and y. Let A be $(\frac{a}{2}, y)$. Tension of chain at $A=\omega y$ and this tension supports the weight of the chain AN, which is $\omega.AN$.

$\therefore\ \omega y=\omega.AN\quad or,\quad AN=y\quad \therefore\ N$ is on Ox.

Similarly M is on Ox. l = length of whole chain $=2s+2y$, where $s=CA$.

Considering the point A and using the formula $s+y=ce^{\frac{x}{c}}$,

we get, $\frac{l}{2}=ce^{\frac{a}{2c}}$......(1)

This equation gives l as a function of c.

For extreme values of l, we have, $\frac{dl}{dc}=2e^{\frac{a}{2c}}\left[1-\frac{a}{2c}\right]=0$

$\therefore\ c=\frac{a}{2}$; again, $\frac{d^2l}{dc^2}=2e^{\frac{a}{2c}}\cdot\frac{a^2}{4c^3}>0$.

$\therefore$ $c=\frac{a}{2}$ gives the minimum value of l which is ae [from (1)].

$\therefore$ for equilibrium to be possible, we must have $l \geqslant ae$.

Ex. 10. A uniform chain of length s, hangs from two fixed points A, B on the same level at a distance $2a$ apart. Show that if s varies, the minimum depth of the directrix of the catenary is $\frac{a}{\sqrt{z^2-1}}$, where z is given by $z \tanh z=1$. [C. H. 1968].

Let Ox be the directrix of the catenary (see fig. Ex. 1, p. 205). If y be the depth of Ox below AB, then height of A above Ox is also y and $\therefore$ A is (a, y). Since $y=c \cosh \frac{x}{c}$ and $s=c \sinh \frac{x}{c}$, we have, here, $y=c \cosh \frac{a}{c}$ and $\frac{s}{2}=c \sinh \frac{a}{c}$.

From 2nd eqn. it follows that as s varies c varies. $\therefore$ we have to find the minimum value of y when c varies.

Differentiating y w. r. to c, we get,

$$\frac{dy}{dc}=\cosh \frac{a}{c}-\frac{a}{c} \sinh \frac{a}{c} \ldots\ldots(1)$$

and $$\frac{d^2y}{dc^2}=-\frac{a}{c^2} \sinh \frac{a}{c}+\frac{a}{c^2} \sinh \frac{a}{c}+\frac{a}{c} \cosh \frac{a}{c} \cdot \frac{a}{c^2}$$

$$=\frac{a^2}{c^3} \cosh \frac{a}{c}>0 \text{ for all } +ve\ c.$$

For max. or min. values of y, we have, $\frac{dy}{dc}=0$.

or, $\cosh \frac{a}{c}-\frac{a}{c} \sinh \frac{a}{c}=0$ or $\frac{a}{c} \tanh \frac{a}{c}=1$

or, $z \tanh z=1\ldots\ldots(2)$, where $a=cz$.

Eqn. (2) gives values of c for which y is a max. or a min.

Since $\frac{d^2y}{dc^2}>0$ $\therefore$ The value of c in (2) makes y a min. and the minimum value of y is $\frac{a}{z} \cosh z$, where z is given by $z \tanh z=1$. Now $\operatorname{sech}^2 z=1-\tanh^2 z=1-\frac{1}{z^2}$ or $\operatorname{sech} z=\frac{\sqrt{z^2-1}}{z}$.

$\therefore$ Minimum value of the depth of the directrix is

$$\frac{a}{z} \cdot \frac{z}{\sqrt{z^2-1}}=\frac{a}{\sqrt{z^2-1}}.$$

EXAMPLES

1. In a common catenary if T be the tension at any point P, T_0 that at the lowest point C, and W be the weight of the portion CP of the string show that $T^2 - T_0{}^2 = W^2$.

2. A uniform chain of length $4a$ is suspended from two points A and B in the same horizontal line and the tension at A is $\frac{5}{3}$ times the tension at the lowest point. Prove that the span AB is $3a \log 3$.

3. A heavy uniform chain of length l is suspended from two fixed points A and B in the same horizontal line, and the tangent at A makes an angle of 45° with the horizon. If k is the sag in the middle prove that $2k = (\sqrt{2} - 1)l$.

4. The extremities A and B of a heavy string of length $2l$ are attached to two small rings which can slide on a fixed horizontal wire. Each of these rings is acted on by a horizontal force ωl. Show that the distance apart of the rings is $2l \log (1 + \sqrt{2})$.

5. A uniform chain of length $2l$ is suspended from two points in the same horizontal line distant $2a$ apart. If the sag in the middle be a and the parameter be c, show that $\frac{2a^2}{l^2 - a^2} = \log \frac{l + a}{l - a}$, $\tanh \frac{a}{c} = \frac{2al}{a^2 + l^2}$.

6. A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite the inclination of the wire to the ground is $2 \tan^{-1} \left(\frac{h}{l}\right)$, and that its tension there and at the ground are $\omega . \frac{l^2 + h^2}{2h}$ and $\omega . \frac{l^2 - h^2}{2h}$, where ω is the weight of the wire per unit of length.

7. A length of uniform chain hangs over two or more smooth pegs in a vertical plane. Prove that the catenaries formed by the various portions have the same directrix and that the free ends of the chain lie on the directrix.

8. A heavy uniform string hangs over two smooth pegs in the same horizontal line. If the length of each portion which hangs freely be equal to the length between the pegs, prove that the whole length of the string is to the distance between the pegs as $\sqrt{3}$ to $\log (\sqrt{3})$.

9. A heavy uniform string 42 ft long hangs over two smooth pegs at different heights. The portions which hang vertically are 15 ft. and 13 ft. in length. Prove that the parameter of the catenary is 12 ft., that its vertex divides the string in the ratio 3 : 4 and that the horizontal and vertical distances between the pegs are 12 log 3 and 2 feet.

10. A heavy uniform string, of length l is suspended from a fixed point A, and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distances between A and B are $a \sinh^{-1} \frac{l}{a}$ and $(\sqrt{l^2 + a^2} - a)$.

11. The end links of a uniform chain can slide on a fixed rough horizontal rod. Prove that the ratio of the extreme span to the length of the chain is $\mu \log \dfrac{1+\sqrt{1+\mu^2}}{\mu}$, where μ is the coefficient of friction. [C. H. 1952]

12. A uniform chain of length l and weight W hangs between two fixed points at the same level, and a weight W' is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of support is $\dfrac{k}{2l}W+\dfrac{l}{4k}W'+\dfrac{l}{8k}W$. [C. H. 1955]

13. A heavy string of uniform density and thickness is suspended from two fixed points in the same horizontal line. A weight equal to $\dfrac{1}{n}$th. of the weight of the string is attached to its lowest point. If θ, ϕ be the inclinations to the vertical of the tangents at the highest and lowest points of the string, prove that $\tan\phi=(n+1)\tan\theta$. [C. H. 1966]

14. A uniform string of given length is suspended between two points in the same horizontal line. Show that when the sag in the middle decreases, the terminal tension increases. If $2l$ be the length of the string and if the string cannot bear a tension greater than $b\omega$, where ω is the weight per unit length of the string, show that the sag cannot be less than $(b-\sqrt{b^2-l^2})$.

15. A uniform chain hangs over two fixed pegs in a horizontal line, distant $2a$ apart, its extremities being free; find the parameter of the catenary when the pressure on each peg is $b\omega$, ω being the weight of unit length of the string.

16. A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of same height, as will produce the least possible tension at the posts. Show that $l=\dfrac{d}{\lambda}\sinh\lambda$, where λ is given by the equation $\lambda\tanh\lambda=1$.

17. One end of a uniform chain ABC of length l is attached to a fixed point A at a height h above a rough table. The portion BC is straight and rests on the table in a vertical plane through A. If the end C be free, prove that in limiting equilibrium the length z of the hanging portion is given by the equation $z^2+2\mu hz=h^2+2\mu hl$. Also find the parameter of the catenary.

18. A uniform chain of length l hangs between two points A and B at a horizontal distance a apart with B at a vertical distance b above A. If T_1, T_2 be the tensions at A and B, show that

$$T_1+T_2=W\sqrt{\left(1+\frac{4c^2}{l^2-b^2}\right)}.$$

19. A kite is flown with 600 ft of string from the hand to the kite, and a spring balance held in the hand shows a pull equal to the weight of 100 feet of the string, inclined at 30° to the horizontal. Find the vertical height to the kite above the hand. [Lond. B. Sc.]

20. In any net work of strings hanging in a vertical plane, if three uniform strings of the same line density meet in a knot, and their directions at the point make angles θ, ϕ, ψ with the horizontal, prove that the parameters of the catenaries are numerically as $\tan\phi - \tan\psi : \tan\psi - \tan\theta : \tan\theta - \tan\phi$.

6·5. Determination of the parameter.

To determine the parameter of the catenary in which a uniform string of given length hangs when suspended between two given points.

Let A and C be the two given points and let l be the length of the string suspended between A and C (see fig. of Ex. 7. p. 210) ; let the co-ordinates of A relative to C be (h, k). Let the parameter of the catenary in which the string hangs be c and let the co-ordinates of C, referred to Ox, OD the directrix and axis of the catenary be (x, y). Then those of A are $(x+h, y+k)$. Using the formulæ $y=c\cosh\frac{x}{c}$ and $s=c\sinh\frac{x}{c}$ for the points A and C, we have, $y+k=c\cosh\frac{x+h}{c}$, $s_0+l=c\sinh\frac{x+h}{c}$

and $y=c\cosh\frac{x}{c}$, $\qquad s_0=c\sinh\frac{x}{c}$, where s_0 = length of the catenary from the vertex D to the point C.

$$\therefore\ k=c\left[\cosh\frac{x+h}{c}-\cosh\frac{x}{c}\right] \text{ and } l=c\left[\sinh\frac{x+h}{c}-\sinh\frac{x}{c}\right]$$

$$\therefore\ l^2-k^2=c^2\left[2\cosh\frac{x+h}{c}\cosh\frac{x}{c}-2\sinh\frac{x+h}{c}\sinh\frac{x}{c}-2\right]$$

$$=2c^2\left[\cosh\frac{h}{c}-1\right]=4c^2\sinh^2\frac{h}{2c}.$$

$$\therefore\ 2c\sinh\frac{h}{2c}=\pm\sqrt{l^2-k^2}\cdots\cdots(1)$$

This equation gives the value of c. It can be shown that this equation gives *only one* real positive value of c.

Put $\frac{h}{2c}=\xi$. Then equation (1) becomes $\frac{\sinh\xi}{\xi}=\pm m$,

where $m=\frac{\sqrt{l^2-k^2}}{h}>1$, as $l>\sqrt{k^2+h^2}$. Since here ξ is $+ve$.

$\therefore\ \frac{\sinh \xi}{\xi}$ is $+ve$. $\therefore$ Our equation for determining c or ξ is $\frac{\sinh \xi}{\xi}=m$. For $\xi>0$, $\frac{\sinh \xi}{\xi}=f(\xi)$ is a positive strictly monotonic increasing continuous function of ξ and $\underset{\xi\to 0}{\text{Lt}}\ f(\xi)=1$, and $f(\xi)\to\infty$ as $\xi\to\infty$. Since $m>1$, from the properties of a continuous function it follows that there is one and only one real $+ve$ value of ξ for which $f(\xi)=m$. Hence $c=\frac{h}{2\xi}$ has one and only one real positive value. So the parameter of the catenary is definitely determinate.

6·6. *If a heavy uniform string suspended from two points (not in same vertical) be nearly straight, then the parameter of the catenary in which it hangs is very large.*

Let A, B be the ends of the string and C the vertex of the catenary in which it hangs. Let ψ, ψ' be the angles the tangents at A and B make with the horizontal. Then $CA=c\tan\psi$, $CB=c\tan\psi'$ where c is the parameter.

$$\therefore\quad l=CA-CB=c\,(\tan\psi-\tan\psi')$$

$$\therefore\quad c=\frac{l}{\tan\psi-\tan\psi'}.$$

As ψ is very nearly equal to ψ', it follows that c is very large compared to l, the length of the string.

6·7. *To find the parameter and the sag in the middle when a uniform string is tightly stretched between two points in the same horizontal line.*

Let A, B be the two points in the same horizontal line from which the string ACB is suspended, whose lowest point is C (see fig. Ex. 1, p. 205). The depth of C below AB is called the *sag* in the middle. Let the sag be k. Let AB be a and let l be the length of the string ; also let c be the parameter. Using the formula $s=c\sinh\frac{x}{c}$ for the point A, we have,

$$\frac{l}{2}=c\sinh\frac{a}{2c}. \quad or,\quad \frac{l}{a}=\frac{\sinh\xi}{\xi}\ \text{where}\ \xi=\frac{a}{2c}.$$

Since the string is tightly stretched $\frac{l}{a}$ is very nearly equal to 1 and so ξ is nearly zero $\therefore$ c is very large compared to a

or l. Since c is very large (this also follows from previous article), expanding $\sinh \frac{a}{2c}$, and neglecting terms of higher order of smallness, we have $l=2c\left[\frac{a}{2c}+\frac{a^3}{48c^3}\right]$

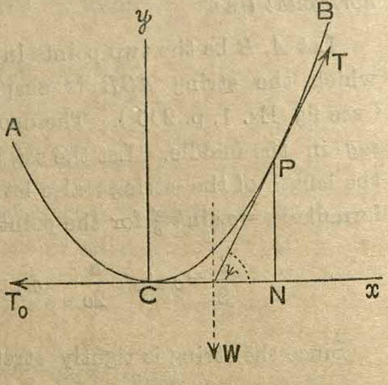
or, $c^2=\frac{a^3}{24(l-a)}$. This gives the value of the parameter when a and l are given. To find the sag k, we use the formula $y=c\cosh\frac{x}{c}$ for the point A and get $(c+k)=c\cosh\frac{a}{2c}=c\left[1+\frac{a^2}{8c^2}\right]$, neglecting terms of higher order of smallness.

$\therefore$ $k=\frac{a^2}{8c}=\frac{1}{4}\sqrt{6a(l-a)}$, in terms of a, l We can express the sag in terms of the maximum tension which the string can bear. If T be the maximum tension, which is the tension at A, we have $T=\omega(c+k)=\omega c$, approx. as c is very large and k is very small.

$\therefore$ we have, $k=\frac{a^2}{8c}=\frac{a^2\omega}{8T}$ *or*, $k=\frac{l^2\omega}{8T}$, as l is nearly equal to a.

6·8. Parabolic Chain—*A chain suspended from two points supports a continuous load so that the load supported by any element of the chain is proportional to the horizontal projection of the element. The weight of the chain being neglected, show that the chain hangs in the form of a parabola.*

Let the origin be taken at C, the lowest point of the chain and let the horizontal tangent at C and the vertical through C be taken as axes of x and y. Let $P(x, y)$ be any point of the chain and consider the separate equilibrium of the portion CP of the chain. It is in equilibrium under the action of three forces viz. the tension T at P, the the tension T_0 at C and the weight W of the load supported by the portion CP. Now $W \propto CN$, where PN



is ⊥ on Cx and $CN=x$. $\therefore$ $W=kx$, where k is a constant. If ψ be the angle T makes with Cx, we get, resolving the forces horizontally and vertically,

$$T\cos\psi=T_0 \quad \text{and } T\sin\psi=kx.$$

Dividing one by the other, we have, $\tan\psi=\frac{k}{T_0}x=2ax$,

where $a=\frac{k}{2T_0}$ $\therefore$ $\frac{dy}{dx}=2ax$. Integrating $y=ax^2+A$.

At C, $x=0$, $y=0$ $\therefore$ $A=0$ $\therefore$ $y=ax^2$.

This is the equation of the curve in which the chain hangs, and we know it is the equation of a parabola whose vertex is C and axis is Cy.

In a *Suspension bridge* we have two parallel chains supporting the horizontal roadway of the bridge by means of vertical rods, the horizontal distances between consecutive rods being equal. If the number of rods be assumed to be very large and the weights of the chains and the rods be negligible compared to the weight of the roadway, the problem of the suspension bridge reduces to that of the parabolic chain.

6·9. Catenary of uniform Strength.

Now we shall consider the case of *a heavy string, not uniform but heterogeneous, hanging freely in equilibrium, being suspended from two fixed points, the cross-section at any point of the string being proportional to the tension at that point.*

Take the origin at C, the lowest point of the string (fig. of Art. 6·8). and take the horizontal tangent at C as the x-axis and the vertical through C as the y-axis. Let $P(x, y)$ be any point of the string and consider the separate equilibrium of the portion CP of the string. It is in equilibrium under three forces, the tension T at P, the tension T_0 at C and the weight W of the portion CP. If T makes an angle ψ with Cx, we have, resolving horizontally and vertically.

$$T\cos\psi=T_0, \quad T\sin\psi=W\ldots\ldots(1)$$

Let the arcual distance be measured from C and let w be the weight per unit length of the string at a point whose arcual distance is s.

Then $W=\int_0^s w\,ds$. where arc $CP=s$.

$\therefore$ from (1), we have, $T_0 \tan\psi = \int_0^s \omega ds.$

Differentiating with respect to s, we get $T_0 \sec^2\psi \dfrac{d\psi}{ds} = \omega$...(2)

So far the treatment is perfectly general and is applicable to the problem of any heterogeneous string in equilibrium under gravity. Now we have to find ω in the present case.

Let ρ = uniform density of the string.

$\sphericalangle$ = cross-section of the string at any point Q where arc CQ is s.

ω = weight per unit length at Q.

Take a small element of the string of length δs containing the point Q.

Then weight of the element $= \sphericalangle . \delta s . \rho . g$.

$$\therefore \quad \omega = \lim_{\delta s \to 0} \frac{\sphericalangle\, \delta s . \rho . g}{\delta s} = \sphericalangle \rho g.$$

It is given that $\sphericalangle \propto T \quad \therefore \quad \sphericalangle = kT$, where k is a constant

$\therefore \quad \omega = kTg\rho \propto T$. Let $\omega = \dfrac{T}{a}$, where a is a constant.

From (2), we have, $T_0 \sec^2\psi \dfrac{d\psi}{ds} = \omega = \dfrac{T}{a} = \dfrac{T_0}{a} \sec\psi.$

$$\therefore \quad \frac{d\psi}{ds} = \frac{1}{a} \cos\psi = \frac{1}{a} . \frac{dx}{ds}.$$

Integrating, we get $\psi = \dfrac{x}{a} + A.$

At C, $x=0$, $\psi=0 \quad \therefore \quad A=0. \quad \therefore \quad x = a\psi$......(3)

$$\therefore \quad \frac{dy}{dx} = \tan\psi = \tan\frac{x}{a}.$$

Integrating, $y = a \log\sec\dfrac{x}{a} + B.$

At C, $x=0$, $y=0 \quad \therefore \quad B=0 \quad \therefore \quad y = a \log\sec\dfrac{x}{a}$......(4)

This is the Cartesian equation of the curve in which the string hangs. It is known as the **Catenary of Uniform Strength**, so called by Davies Gilbert, who invented it. The constant a is called the *parameter* of the catenary.

The curve is symmetrical about the y-axis for its eqn. is not altered by writing $-x$ for x. Again as $\frac{x}{a}$ increases from 0 to $\frac{\pi}{2}$, $\sec\frac{x}{a}$ increases from 1 to ∞ and so y increases from 0 to ∞. When $\frac{x}{a}$ becomes $>\frac{\pi}{2}$, $\sec\frac{x}{a}$ becomes $-ve$ and so y becomes imaginary.

$\therefore$ The curve has two vertical asymptotes $x=\pm\frac{\pi a}{2}$ between which it wholly lies. The *span* of the curve cannot, therefore, be greater than πa.

We have seen that $\propto\infty T$ gives $\omega\infty T$. $\therefore$ $\omega\propto T$ may be taken to give the law of linear density for the Catenary of Uniform strength.

Examples Worked

Ex. 1. A telegraph is constructed of iron wire which weighs 7·3 lbs. per 100 feet; the distance between the posts is 150 feet and the wire sags 1 foot in the middle. Show that it is screwed up to a tension of about 205 lbs. wt.

Here $\omega=\frac{7\cdot3}{100}=\cdot073$ lbs. wt.

Let c be the parameter of the catenary in feet, then for an end of the wire, we have $x=75$ ft. and $y=(1+c)$ ft.

$\therefore$ Using the formula, $y=c\cosh\frac{x}{c}$,

We have, $1+c=c\cosh\frac{75}{c}=c\left[1+\frac{75^2}{2.c^2}+\cdots\right]$

Since the sag is very small compared to the distance between the posts, it shows that the string is tightly stretched and therefore c is very large.

$\therefore$ $1+c=c+\frac{75^2}{2c}$ approximately

This gives $c=\frac{75^2}{2}$. ,,

$\therefore$ Tension at a point of suspension $=\omega y=\omega(1+c)$

$$=\cdot073\left(1+\frac{5625}{2}\right)\text{ lbs. wt.}=205\text{ lbs. wt. nearly.}$$

or directly using the result of the article 6·7,

$$T=\frac{a^2\omega}{8k}=\frac{(150)^2.7\cdot3}{8\times100}=205 \text{ lbs. wt. nearly.}$$

Ex. 2. A wire 140 yds. long hangs between two points, 138 yds. apart horizontally and 50 ft. vertically ; show that the tension at its lowest point is about 495 lbs. wt., the wire weighing half a pound per foot.

We shall use the formula $2c\sinh\frac{h}{2c}=\sqrt{l^2-k^2}$ (See Art 6·5)

Here $l=140\times3$ ft., $h=138\times3$ ft., $k=50$ ft.,

∴ $\sqrt{l^2-k^2}=10\sqrt{1739}=10\times41\cdot7$ nearly $=417$.

Since c is very large, we have $2c\left[\frac{h}{2c}+\frac{h^3}{48c^3}\right]=417$

or, $\frac{h^3}{24c^2}=417-414=3.$ ∴ $c^2=\frac{(138\times3)^3}{72}$

∴ $c=207\sqrt{23}=992\cdot5$ nearly ∴ $T_0=c.\frac{1}{2}=496$ nearly

∴ tension reqd. $=496$ lbs wt. nearly.

Ex. 3. A uniform chain of length l when tightly stretched between two points at the same level has a sag k in the middle. Prove that the length of the chain exceeds the distance between the points by $\frac{8k^2}{3l}$. [C. H. 1944]

Let l be the length of the chain and a the distance between the two points, then we have,

$$l-a=\frac{a^3}{24c^2} \text{ and } k=\frac{a^2}{8c} \quad \text{[See Art 6·7]}$$

∴ $l-a=\frac{a^3}{24.a^4}\cdot64k^2=\frac{8k^2}{3a}=\frac{8k^2}{3l}$, as l is nearly equal to a.

Ex. 4. A telegraph wire weighing 1 oz. per foot is stretched nearly horizontal between two points 100 yds. apart. Find the sag in the middle if the tension is not to exceed 150 lbs.

Here $a=100\times3$ ft., $\omega=\frac{1}{16}$ lbs. per foot, $T=150$ lbs. wt.

∴ $k=\frac{a^2\omega}{8T}=\frac{9\times10^4}{16\times8\times150}=4\cdot68$ ft.

Ex. 5. The span of a telegraph wire is 200 ft. and the length of the wire is 202 ft. Find the sag in the middle. [C. H. 1942]

Here $a=200$, $l=202$

$\therefore \quad k=\frac{1}{4}\sqrt{6a(l-a)}=\frac{1}{4}\sqrt{6\times 200\times 2}=5\sqrt{6}$ ft. [See Art. 6·7]

6·10. Determination of the Law of Density.

A heterogeneous string is in equilibrium under the action of gravity alone. The form assumed by the string when in equilibrium depends on the law of density and for different distribution of density the curve of equilibrium will be different. If the equation of the curve of equilibrium be given, we can find the law of density of the string *i.e.* ω. For from (2) of the previous article we have, $\omega=\dfrac{T_0}{\rho\cos^2\psi}$, where ρ is the radius of curvature. If the equation of the curve is given in the intrinsic form $s=f(\psi)$, then $\rho=f'(\psi)$ and $\omega=\dfrac{T_0}{f'(\psi)\cos^2\psi}=\phi(s)$. This gives the density at a point whose arcual distance is s. If the equation is given in the Cartesian form $y=f(x)$, we have, $\tan\psi=\dfrac{dy}{dx}=f'(x)$. Also ρ can be found in terms of x. Therefore ω is expressed in terms of x. We can also express ω in terms of y, if more convenient. These give the density at a point of the string when its abscissa or ordinate is known.

Examples Worked

Ex. 1. The whole load of a suspension bridge is 200 tons evenly distributed over its horizontal span, which is 150 ft. and its height is 20 ft. Show that the tensions at the lowest point and at the points of support are 187·5 and 212·5 tons weight respectively.

Take the lowest point O as the origin and the tangent at O and the vertical through it as axes of x and y. Let P be a point of support and let the equation of the parabolic chain be $x^2=4ay$. Since co-ordinates of P are (75, 20), we have, $4a=\dfrac{(75)^2}{20}=\dfrac{75\times 15}{4}$.

If the tangent at P makes an angle ψ with Ox, then

$$\tan\psi=\frac{dy}{dx}=\frac{x}{2a}=\frac{75\times 8}{75\times 15}=\frac{8}{15} \quad \therefore \quad \sin\psi=\frac{8}{17},\ \cos\psi=\frac{15}{17}.$$

Let T, T_0 be the tensions in tons weight at P and O. Considering equilibrium of the portion OP of the chain which supports a load $=\frac{200}{2}=100$ tons weight, we have,

$$T\cos\psi=T_0 \text{ and } T\sin\psi=100.$$

$$\therefore\quad T=\frac{100\times17}{8}=\frac{25\times17}{2}=212{\cdot}5$$

$$\text{and } T_0=\frac{25\times17}{2}\times\frac{15}{17}=\frac{25\times15}{2}=187{\cdot}5.$$

Hence required tensions are 187·5 and 212·5 tons weight.

Ex. 2. Prove that in the parabolic catenary the tension varies as the square root of the height above the directrix.

Referred to the horizontal and the vertical through the lowest point as axes the equation of the parabolic chain is

$$y=ax^2\text{......(1)} \quad [\text{ See Art. } 6{\cdot}8\ ]$$

Also T being the tension at a point P (x, y), we have,

$$T\cos\psi=T_0\text{......(2)}$$

where ψ is the angle the tangent at P makes with Ox.

$$\text{Now}\quad \tan\psi=\frac{dy}{dx}=2ax, \quad [\text{differentiating (1)}$$

$$\therefore\quad \sec^2\psi=1+\tan^2\psi=1+4a^2x^2=1+4ay \quad [\text{ from (1) }]$$

$$=4a\left(\frac{1}{4a}+y\right).$$

Latus rectum of the parabola (1), which is $x^2=\frac{1}{a}y$, is $\frac{1}{a}$.

$\therefore$ Depth of directrix below $Ox=\frac{1}{4a}$

$\therefore$ $y+\frac{1}{4a}$ is the height of P above the directrix $=h$, say

$$\therefore\quad T=T_0\sec\psi=T_0\sqrt{4ah}.$$

Hence $T\propto\sqrt{\text{height above directrix.}}$

Ex. 3. Prove that in a catenary of uniform strength, $s=a\log(\sec\psi+\tan\psi)$, $\rho=a\sec\psi=a\cosh\frac{s}{a}$. Hence show that the mass per unit length at any point varies as the corresponding radius of curvature.

Eqn. of the catenary is $y = a \log \sec \dfrac{x}{a}$

$$\therefore \quad \tan\psi = \frac{dy}{dx} = \tan\frac{x}{a} \quad \therefore \quad \psi = \frac{x}{a} + n\pi.$$

When $x=0$, $\psi=0$; $\therefore \; n=0$. $\quad \therefore \; x = a\psi$.

$$\therefore \quad \cos\psi = \frac{dx}{ds} = \frac{dx}{d\psi}\cdot\frac{d\psi}{ds} = a\frac{d\psi}{ds} \qquad \therefore \; \rho\cos\psi = a \ldots(1)$$

Also $ds = a \sec\psi \, d\psi$ $\quad \therefore \; s = a \log(\sec\psi + \tan\psi) + B$

When $s=0$, $\psi=0$; $\quad \therefore \; B=0$.

Hence $s = a \log(\sec\psi + \tan\psi) \ldots(2)$

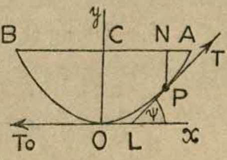
From (2) $\sec\psi + \tan\psi = e^{\frac{s}{a}}$ $\quad \therefore \; \sec\psi - \tan\psi = e^{-\frac{s}{a}}$

Adding, $\sec\psi = \cosh\frac{s}{a}$ $\;\therefore\;$ From (1) $\rho = a\sec\psi = a\cosh\frac{s}{a}$

We have, $\omega = \dfrac{T_0}{\rho\cos^2\psi} = \dfrac{T_0\rho^2}{\rho a^2}$ from (1) $= \dfrac{T_0}{a^2}\rho \propto \rho$

$$\therefore \quad \text{mass per unit length} \propto \rho \propto \cosh\frac{s}{a}.$$

Ex. 4. Find the mass per unit length of a chain which hangs under gravity in the form of a semi-circle of radius a with its bounding diameter horizontal. [C. H. 1967]



The chain is suspended from A, B where AB is horizontal. We take the origin at the lowest point O and the tangent Ox at O as the x-axis.

The equation of the semi-circle is $s = a\psi$ where s is the arc OP measured from O to any point P and ψ is the angle the tangent PL at P makes with Ox. Let T_0, T be the tensions at O and P respectively and let m be the mass per unit length at P. Considering equilibrium of the portion OP of the chain, and resolving horizontally and vertically, we have,

$$T\cos\psi = T_0, \; T\sin\psi = \int_0^s mg\,ds.$$

$$\therefore \quad T_0\frac{d}{ds}(\tan\psi) = mg. \quad \therefore \quad mg = T_0 \sec^2\psi\frac{d\psi}{ds} = \frac{T_0}{a}\sec^2\psi$$

$$\therefore \quad m = \frac{T_0}{ag}\sec^2\psi = \frac{T_0}{ag}\sec^2\frac{s}{a}, \text{ in terms of } s.$$

We have $\angle OCP = \psi$. If PN be the perpendicular from P on AB, then $PN = a \cos\psi$.

$\therefore \; m = \dfrac{T_0 a}{gPN^2} \propto \dfrac{1}{PN^2}$ $\quad\therefore$ We may also say that the mass per unit length at any point varies inversely as the square of its depth below AB.

Ex. 5. A chain hangs in the form of the semi-circle $y = a - \sqrt{a^2 - x^2}$, the axis of y being vertical. If T_0 be the tension at the lowest point, find the weight per unit length of the chain in terms of y. [C. H. 1962, 1969]

The equation is $(y-a)^2 = a^2 - x^2$ *or*, $x^2 + (y-a)^2 = a^2$, a circle whose centre is $(0, a)$ and radius is a. It is a circle which touches the x-axis at the origin on the positive side. Hence the origin O is the lowest point of the chain and the horizontal tangent at O is the x-axis, the y-axis being upwards.

[See fig. of Ex. 4 above]

Differentiating the given equation,

$$\tan\psi = \frac{dy}{dx} = \frac{x}{\sqrt{a^2 - x^2}} \quad \therefore \quad \sec^2\psi = \frac{a^2}{a^2 - x^2}.$$

Radius of curvature $\rho = a$.

$\therefore$ Using the result, $\omega = \dfrac{T_0}{\rho \cos^2\psi}$,

we have, $\omega = \dfrac{T_0 a}{a^2 - x^2} = \dfrac{T_0 a}{(a-y)^2}$. This is the same sum as Ex. 4 worked above. Here the equation given is Cartesian.

Ex. 6. Find the form in which a chain hangs when the line-density is given by $\dfrac{T_0}{ga} \sec^2 \dfrac{s}{a}$, T_0 being the tension at the lowest point, and s being measured from this point. [C. H. 1957, 1964]

We take the origin at the lowest point O and the tangent Ox at O as the x-axis (See fig. of Ex. 4 above). If the tension at P, where arc $OP = s$, be T making an angle ψ with Ox, we consider the equilibrium of the portion OP of the chain and resolving horizontally and vertically, we have, $T\cos\psi = T_0\ldots(1)$

and
$$T\sin\psi = \int_0^s \omega\, ds = \int_0^s \left(\frac{T_0}{ga}\sec^2\frac{s}{a}\right) g\, ds$$
$$= T_0\left[\tan\frac{s}{a}\right]_0^s = T_0\tan\frac{s}{a}\ldots\ldots(2)$$

From (1) and (2) we have $\tan\psi = \tan\frac{s}{a}$ $\therefore$ $\psi = \frac{s}{a}$, as $\psi = 0$ when $s = 0$, *or*, $s = a\psi$, which is the intrinsic equation of a *circle* whose radius is a and whose centre is on y-axis at a distance a from origin.

EXAMPLES

1. A telegraph wire has a span of 80 ft and the wire weighs 1lb per 20 ft; find the sag in the middle if the tension is not to exceed 160 lbs.

2. Show that the maximum tension in a wire which weighs ·15 lb per yard and hangs with a sag of 1 foot in a horizontal span of 100 ft is about 62·5 lbs. wt.

3. A uniform rope weighs 2 lbs per foot; it is tightly stretched between two points in the same level 150 ft. apart. If the sag in the middle is 5 ft., show that the length of the rope is 150·44 ft. and the tension is 1125 lbs.

4. A uniform chain, 155 ft. long, is suspended from two points in the same horizontal line 150 ft. apart. Show that the least tension is nearly 1·08 times the weight of the chain.

5. Prove that in a parabolic chain the tension at any point is $\left(\frac{T_0}{2a}\right)$ times the length of the normal between the point and the axis of the parabola, where $2a$ is the semi-latus rectum.

6. The span of a suspension bridge is 100 ft. and the sag at the middle of the chain is 10 ft.; if the load on each chain be 25 tons, find the greatest tension in each chain and the tension at the lowest point.

7. Show that the tensions at the ends of a chain whose span is 30 ft., the sag in the middle is 10 ft., and the load, which is uniformly distributed across the span, half a ton per foot run of the span are each 9·375 tons wt.

8. Prove that in a catenary of uniform strength, (*a*) the tension varies as the radius of curvature, (*b*) the projection of the radius of curvature on the vertical is constant and (*c*) $T = T_0\cosh\frac{s}{a}$. [C. H. 1945]

9. If the line density at any point of a cord vary as the radius of curvature of the curve in which it hangs, show that this curve is the catenary of uniform strength.

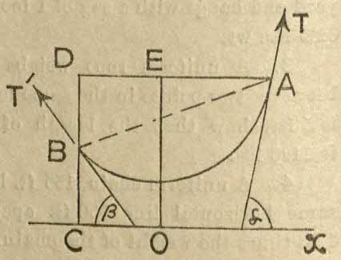
10. Find the equation of the curve if the load supported by a string is ax^2+b per horizontal foot, where a, b are constants and x is the horizontal distance measured from the lowest point of the string. [C. H. 1963]

11. Find the law of variation of mass per unit length in order that a string may hang in the form of a parabola. [C. H. 1942]

12. In a chain hanging in the form of a cycloid whose equation is $s=4a\sin\psi$, show that the mass per unit length is proportional to $(16a^2-s^2)^{-\frac{3}{2}}$. [C. H. 1965]

Miscellaneous Problems Worked

Ex. 1. The ends of a uniform heavy chain are attached to two fixed points A, B. When the chain is hanging in equilibrium the tangents to it at its extremities make angles $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{5}{12}$ with the horizontal in opposite senses. Prove that the inclination of the straight line AB to the horizontal is $\cot^{-1}(6\log 3)$.



If α, β be the angles the tangents at A and B make with the directrix Ox, we have,

$\tan\alpha=\frac{3}{4}$, $\tan\beta=\frac{5}{12}$.

$\therefore$ $\cos\alpha=\frac{4}{5}$, $\cos\beta=\frac{12}{13}$.

We have (see fig.)

$EA=c\log(\sec\alpha+\tan\alpha)$
$=c\log(\frac{5}{4}+\frac{3}{4})=c\log 2.$

$ED=c\log(\sec\beta+\tan\beta)$
$=c\log(\frac{13+5}{12})=c\log\frac{3}{2}.$

$\therefore$ $DA=DE+EA=c\log\frac{3}{2}+c\log 2=c\log 3.$

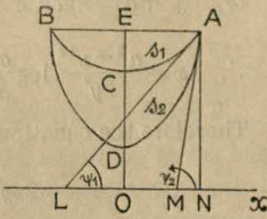
Again, $BC=c\sec\beta=c.\frac{13}{12}$, $DC=c\sec\alpha=c.\frac{5}{4}$.

$$\therefore\ DB=DC-BC=c\cdot\frac{2}{12}=\frac{c}{6}$$

$$\therefore\ \cot BAD=\frac{DA}{DB}=6\log 3 \quad or, \quad \angle BAD=\cot^{-1}(6\log 3).$$

Ex. 2. An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the peg. [M. T.]

Let C, D be the vertices of the two catenaries ACB, ADB formed by the chain when hung over the small smooth pegs A, B. Since the two tensions at A to the two branches are equal and since the tension is ωy, it follows that the two catenaries have the same directrix. Let Ox be the directrix and let the tangents AL, AM at A to the portions CA and DA make angles ψ_1 ane ψ_2 with Ox and let the arcs CA, DA be s_1 and s_2. If c_1, c_2 be the parameters of the catenaries ACB and ADB, we have, $c_1 = CO$, $c_2 = DO$.



$$\therefore \quad CD = c_1 - c_2 \cdots\cdots(1)\ ; \quad s_1 = c_1 \tan\psi_1,\ s_2 = c_2 \tan\psi_2$$

$$\therefore \quad l = s_1 + \varepsilon_2 = c_1 \tan\psi_1 + c_2 \tan\psi_2 \cdots\cdots(2),$$

where $2l$ = length of the chain. Also from ordinate of A, we have, $c_1 \sec\psi_1 = c_2 \sec\psi_2 \cdots\cdots(3)$

$$\therefore \quad \frac{c_1}{\cos\psi_1} = \frac{c_2}{\cos\psi_2} = \frac{c_1 - c_2}{\cos\psi_1 - \cos\psi_2} = \frac{c_1\tan\psi_1 + c_2\tan\psi_2}{\sin\psi_1 + \sin\psi_2}$$

$$or, \quad CD = c_1 - c_2 = l \cdot \frac{\cos\psi_1 - \cos\psi_2}{\sin\psi_1 + \sin\psi_2} = l . \tan\frac{\psi_2 - \psi_1}{2} \quad \text{[From (2)]}$$

$$\therefore \quad CD : l = \tan\left(\tfrac{1}{2}\angle LAM\right).$$

Ex. 3. One end A of a uniform string of length $2a$ is fixed, the other end B moves along a horizontal line through A. Find the locus of the vertex of the catenary formed by the string. [M. T.]

Let AB be $2x'$ and let C the vertex of the catenary, be at a depth y' below AB. Then C is (x', y') referred to horizontal and vertical through A as axes. Referred to the directrix and axis of the catenary as axes B is $(x', y'+c)$, where c is the parameter. Using the formula $y^2 = c^2 + s^2$ for the point B, we have, $(y'+c)^2 = c^2 + a^2$ or, $c = \dfrac{a^2 - y'^2}{2y'} \cdots\cdots(1)$

Again, using the formula, $y + s = ce^{\frac{x}{c}}$ for the point B, we have,

$$(y'+c) + a = ce^{\frac{x'}{c}} \quad or, \quad y' + \frac{a^2 - y'^2}{2y'} + a = ce^{\frac{x'}{c}}$$

$$or,\ (y'+a)^2=2cy'.e^{\frac{x'}{c}}\quad or,\ e^{\frac{x'}{c}}=\frac{a+y'}{a-y'}\quad [\text{ from (1) }].$$

$$\therefore\quad x'=\frac{a^2-y'^2}{2y'}\log\frac{a+y'}{a-y'}.$$

Therefore the equation of the locus of the vertex is

$$2xy=(a^2-y^2)\log\frac{a+y}{a-y}.$$

Ex. 4. A uniform flexible string of length $2s$ and weight $2s\omega$ are attached to rings of weights $2s\omega$ and $4s\omega$ which slide on a rough horizontal rail. If the coefficient of friction between the rail and the rings be $\frac{1}{4}$, prove that the greatest distance between the rings when the string hangs in equilibrium is $\frac{3}{2}s\log 3$.
[C. H. 1965]

When the string is in limiting equilibrium, the frictions at the two rings are not both limiting. We have first to decide which friction will attain the limiting value first. Let A, B be the two rings of weights $4s\omega$ and $2s\omega$ respectively, ACB the chain in equilibrium and T_0 the tension at C, the lowest point of the catenary of parameter c and directrix Ox. Let F, F' be the frictions and R, R' the normal reactions on the rings A, B. From the equilibrium of the whole system and of the ring B and portion BC of the chain respectively, we have, resolving horizontally and vertically,

$$R+R'=4s\omega+2s\omega+2s\omega=8s\omega,\quad F=F'$$

and $R'=2s\omega+s\omega=3s\omega,\quad F'=T_0=c\omega.$

$\therefore\quad R=5s\omega.$

So $F=F'$ and $R>R'$ $\quad\therefore\quad \dfrac{F}{R}<\dfrac{F'}{R'}$

$\therefore$ The friction F' at B will first attain the limiting value $\mu R'$.

Now suppose the string is in limiting equilibrium,

$$\therefore\quad F'=\mu R'.$$

$\therefore\quad c\omega=\mu(3s\omega)\quad or,\quad c=3\mu s=\frac{3}{4}s,\quad$ as $\quad\mu=\frac{1}{4}.$

$\therefore$ The formula, $s=c\tan\psi$, gives for the point B,

$$s=\tfrac{3}{4}s\tan\psi\quad or,\quad \tan\psi=\tfrac{4}{3}.$$

Again, using the formula $x=c\log(\sec\psi+\tan\psi)$, for the point B, we get, $\frac{AB}{2}=\frac{3}{4}s\log\left(\frac{5}{3}+\frac{4}{3}\right)=\frac{3}{4}s\log 3$

$$\therefore\quad AB=\frac{3}{2}s.\log 3.$$

Hence the greatest distance between the rings is $\frac{3}{2}s\log 3$.

Ex. 5. A given length, $2s$ of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is $\sqrt{b^2-s^2}\log\frac{b+s}{b-s}$. [C. H. 1956, 1960]

Let the chain be hung between the points A, B where $AB=2x$ and let c be the parameter of the catenary in which the chain hangs (see fig Ex. 1, p. 205). If y be the height of A above the directrix, we have,

$$y=c\cosh\frac{x}{c}\text{ and }s=c\sinh\frac{x}{c}.$$

Also $y^2=c^2+s^2$. Now, $T=\omega y_1$, where T is the tension at height y_1 above directrix. $\therefore$ x remaining constant, T is max. when y_1 is max. i.e. T is max. at A and the max. value is ωy.

$\therefore$ We have to find the maximum value of x subject to the condition that $\omega y\leqslant\omega b$ *or*, $y\leqslant b$.

We have $y+s=ce^{\frac{x}{c}}$, $y-s=ce^{-\frac{x}{c}}$ $\therefore$ $\frac{y+s}{y-s}=e^{\frac{2x}{c}}$.

or, $2x=c\log\frac{y+s}{y-s}=\sqrt{y^2-s^2}\log\frac{y+s}{y-s}$......(1)

We shall show, now, that as x increases y increases.

Differentiating (1) with respect to y, we have,

$$2\frac{dx}{dy}=\sqrt{y^2-s^2}\left(\frac{-2s}{y^2-s^2}\right)+\frac{y}{\sqrt{y^2-s^2}}\log\frac{y+s}{y-s}$$

$$=\frac{1}{\sqrt{y^2-s^2}}\left[y\log\frac{1+\frac{s}{y}}{1-\frac{s}{y}}-2s\right],\ [s\text{ is a constant and }y>s]$$

$$=\frac{1}{\sqrt{y^2-s^2}}\left[y\,2\left(\frac{s}{y}+\frac{s^3}{3y^3}+\cdots+ve\text{ terms}\right)-2s\right]$$

$$=\frac{1}{\sqrt{y^2-s^2}}\times(\text{ sum of }+ve\text{ terms })>0$$

$\therefore \ \dfrac{dy}{dx}>0$ *or*, y increases with x.

$\therefore$ x is max. when y has its max. value b, for $y \leqslant b$.

$\therefore$ Maximum span$=2.x_{max}=\sqrt{b^2-s^2}\log\dfrac{b+s}{b-s}$

[from (1)]

Ex. 6. A uniform chain hangs over two smooth pegs in the same horizontal line, and at a given distance apart ; find the length of the chain when the pressure on each peg is a minimum.

[C. H. 1943, 1945]

Let a be the distance between the pegs A, B (fig. p. 212) ; and when the length of the chain is l, let T be the tension at A making an angle ψ with the directrix. In equilibrium the free ends of the chain are on the directrix Ox. If $2s$ be the length of the chain hanging between A and B and y be the height of A or B above the directrix, we have, $2s+2y=l$ or $s+y=\dfrac{l}{2}$.

Now $s=c\sinh\dfrac{a}{2c}$, $y=c\cosh\dfrac{a}{2c}$

$\therefore \ \dfrac{l}{2}=ce^{\frac{a}{2c}}$.........(1), gives l in terms of c.

The pressure R on each peg is the resultant of T, T acting at an angle $(90^\circ-\psi)=\phi$ say. $\therefore \ R=2T\cos\dfrac{\phi}{2}$.

To find $\cos\dfrac{\phi}{2}$, we have $s=c\tan\psi=c\cot\phi$, $y=c\sec\psi=c\operatorname{cosec}\phi$

$\therefore \ s+y=c\left(\dfrac{\cos\phi}{\sin\phi}+\dfrac{1}{\sin\phi}\right)=c\cot\dfrac{\phi}{2}$.

or $ce^{\frac{a}{2c}}=c\cot\dfrac{\phi}{2}$ *or* $\tan\dfrac{\phi}{2}=e^{-\frac{a}{2c}}$.

$$\therefore \ R=2\times w\times c\cosh\frac{a}{2c}.\frac{e^{\frac{a}{2c}}}{\sqrt{1+e^{\frac{a}{c}}}}=wc\sqrt{e^{\frac{a}{c}}+1}...(2)$$

For stationary values of R, $\dfrac{dR}{dc}=0$

$$\therefore \ \frac{dR}{dc}=w\left[\sqrt{e^{\frac{a}{c}}+1}-\frac{a}{c}.\frac{e}{2\sqrt{e^{\frac{a}{c}}+1}}\right]=0$$

or $e^{\frac{a}{c}}+1=\dfrac{ae^{\frac{a}{c}}}{2c}$ or $e^{\frac{a}{c}}=\dfrac{2c}{(a-2c)}$. ...(3) ; this eqn. gives c.

We have $\dfrac{dR}{dc}=\dfrac{e^{\frac{a}{c}}\left(1-\frac{a}{2c}\right)+1}{\sqrt{e^{\frac{a}{c}}+1}}$

$$\therefore\quad \frac{d^2R}{dc^2}=\frac{e^{\frac{a}{c}}\left(\frac{a}{2c^2}\right)+\left(1-\frac{a}{2c}\right)e^{\frac{a}{c}}.\left(-\frac{a}{c^2}\right)}{\sqrt{e^{\frac{a}{c}}+1}}$$

$$+\left\{e^{\frac{a}{c}}\left(1-\frac{a}{2c}\right)+1\right\}\frac{d}{dc}.\left(\frac{1}{\sqrt{e^{\frac{a}{c}}+1}}\right).$$

When c is a root of eqn. (3), we have,

$\dfrac{d^2R}{dc^2}=\dfrac{e^{\frac{a}{c}}\left(\frac{a}{c}-1\right)\frac{a}{2c^2}}{\sqrt{e^{\frac{a}{c}}+1}}>0$ as otherwise we find from (3) that c will not be real.

$\therefore$ The value of c given by (3), makes R a minimum.

$\therefore$ R is minimum when c is given by $e^{\frac{a}{c}}=\dfrac{2c}{a-2c}$ and the corresponding length of the chain is given by $l=2ce^{\frac{a}{2c}}$.

Ex. 7. A telegraph wire, stretched between two poles at distance a feet apart, sags n feet in the middle ; prove that the tension at the ends is approximately $w\left(\dfrac{a^2}{8n}+\dfrac{7}{6}n\right)$. [C. H. 1951]

Since the wire is stretched n is small and c, the parameter of the catenary, is very large (fig. p. 205). If y be the height above the directrix of the point of support A and T be the tension there we have $y=c+n$, $T=wy$. Also $y=c\cosh\dfrac{a}{2c}$ as the co-ordinates of the point of support A are $\left(\dfrac{a}{2}, y\right)$.

$\therefore$ $c+n=c\cosh\dfrac{a}{2c}$. This is the eqn. to determine c.

Expanding $\cosh\dfrac{a}{2c}$, we get, $c+n=c\left[1+\dfrac{a^2}{\lfloor 2(2c)^2}+\dfrac{a^4}{\lfloor 4.(2c)^4}+\cdots\right]$

or $n=\dfrac{a^2}{8c}+\dfrac{a^4}{24.16c^3}+\cdots$ (1)

Since c is very large, a first approximation of c is given by

$$n=\frac{a^2}{8c} \text{ or } c=\frac{a^2}{8n}$$

For a second approximation, putting this value of c for $\frac{1}{c^2}$ in (1)

$$\text{we get } n=\frac{a^2}{8c}+\frac{a^4}{24.16}\cdot\frac{1}{c}\cdot\left(\frac{8n}{a^2}\right)^2$$

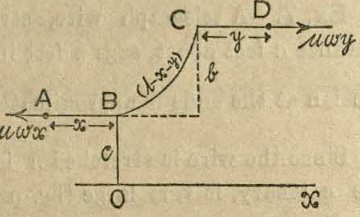
$$=\frac{a^2}{8c}+\frac{n^2}{c.6} \text{ or } c=\frac{a^2}{8n}+\frac{n}{6}.$$

$\therefore$ Approximate value of terminal tension

$$=wy=w(c+n)=w\left(\frac{a^2}{8n}+\frac{7}{6}n\right).$$

Ex. 8. A heavy chain of total length l, rests with one end on a rough table and the other on a rough floor ; it is stretched out as far as possible so that the equilibrium is limiting at both ends ; if μ be the coefficient of friction both of the table and floor and b be the height of the table, show that the length on the floor is $\frac{\mu}{4}\left[b+\frac{2l}{\mu}-\sqrt{b^2+\frac{4lb}{\mu}}\right]-\frac{b}{2\mu}$, provided that $l>b+\frac{b}{\mu}$.

Let $ABCD$ be the chain in limiting equilibrium at both ends A and D. Let $AB=x$ be on floor, $CD=y$ be on the table ; $\therefore$ $BC=l-x-y$ hangs in the form of a catenary. w being the weight per unit length of the chain, friction on AB is μwx and that on CD is μwy. From equilibrium of the point B of BC, it follows that the tension at B of string BC is horizontal. $\therefore$ B is the lowest point of the catenary BC. Let Ox be the directrix and c the parameter, then $BO=c$. From equilibrium of AB, $\mu wx=T_0=cw$. $\therefore$ $c=\mu x\cdots\cdots(1)$ From equilibrium of CD, $\mu wy=T=w(b+c)$. *or*, $\mu y=b+c\cdots(2)$, where T is the tension of BC at C. [The tension T at C acting on the portion BC is along the tangent at C to the catenary, but the tension at C which acts on the portion CD is T along DC *i.e.* horizontal. This is so because at C the table must be assumed to be round and smooth, and to act like a small smooth



peg. This does not alter the magnitude of the tension at C but changes its direction.] Using the formula, $y^2=c^2+s^2$ for the point C, we have, $(b+c)^2=c^2+(l-x-y)^2$

$$\text{or, } b^2+2bc=(l-x-y)^2 \cdots\cdots(3).$$

From (1), (2) and (3), we have to eliminate c and y.

From (1), $c=\mu x$, from (1) and (2), $\mu y=b+\mu x$. or, $y=\frac{b}{\mu}+x$.

$$\therefore\ l-x-y=l-2x-\frac{b}{\mu}=l-2\left(x+\frac{b}{2\mu}\right)$$

$$\therefore\ \text{From (3) } b^2+2b\mu x=\left[l-2\left(x+\frac{b}{2\mu}\right)\right]^2$$

$$\text{or, } 2b\mu\left(x+\frac{b}{2\mu}\right)=\left[l-2\left(x+\frac{b}{2\mu}\right)\right]^2.$$

$$\text{or, } 2b\mu t=(l-2t)^2, \text{ putting } t=x+\frac{b}{2\mu}.$$

Solving, $2t=l+\frac{b\mu}{2}\pm\frac{\mu}{2}\sqrt{b^2+\frac{4lb}{\mu}}$. As $l-x-y=l-2t>0$

$$\therefore\ 2t<l. \quad \therefore\ 2t=l+\frac{b\mu}{2}-\frac{\mu}{2}\sqrt{b^2+\frac{4lb}{\mu}}$$

$$\therefore\ x=\frac{\mu}{4}\left[\frac{2l}{\mu}+b-\sqrt{b^2+\frac{4lb}{\mu}}\right]-\frac{b}{2\mu}.$$

We have assumed $x>0$. Also $l-x-y>b$ [from (3)] and $y>\frac{b}{\mu}$ from (2) $\therefore\ l>b+\frac{b}{\mu}$.

Ex. 9. The distance between the points of support of a catenary of uniform strength is a and the length of the chain is l. Show that the parameter c must be found from

$$\tanh\frac{l}{4c}=\tan\frac{a}{4c}. \qquad \text{[C. H.]}$$

Here it is assumed that the points of support A and B, say, are in the same horizontal line. The equation of the catenary is $y=c\log\sec\frac{x}{c}$. For the point A, we have, $x=\frac{a}{2}$ and $s=\frac{l}{2}$.

Now x and s are both given in terms of ψ by the well known formulae, $x=c\psi$, $s=c\log(\sec\psi+\tan\psi)$.

$\therefore$ We have $\frac{a}{2}=c\psi$ and $\frac{l}{2}=c\log(\sec\psi+\tan\psi)$.

or, $\sec\psi+\tan\psi=e^{\frac{l}{2c}}$

$\therefore$ Eliminating ψ, we have, $\sec\frac{a}{2c}+\tan\frac{a}{2c}=e^{\frac{l}{2c}}$

$\therefore$ $\sec\frac{a}{2c}-\tan\frac{a}{2c}=e^{-\frac{l}{2c}}$ and adding, we have,

$$2\sec\frac{a}{2c}=e^{\frac{l}{2c}}+e^{-\frac{l}{2c}}$$

Adding and subtracting 2 from both sides, we have,

$$2\left(\sec\frac{a}{2c}+1\right)=e^{\frac{l}{2c}}+e^{-\frac{l}{2c}}+2=\left(e^{\frac{l}{4c}}+e^{-\frac{l}{4c}}\right)^2\ldots\ldots(1)$$

and $2\left(\sec\frac{a}{2c}-1\right)=e^{\frac{l}{2c}}+e^{-\frac{l}{2c}}-2=\left(e^{\frac{l}{4c}}-e^{-\frac{l}{4c}}\right)^2\ldots\ldots(2)$

$\therefore$ Dividing (2) by (1), $\dfrac{1-\cos\frac{a}{2c}}{1+\cos\frac{a}{2c}}=\left(\dfrac{e^{\frac{l}{4c}}-e^{-\frac{l}{4c}}}{e^{\frac{l}{4c}}+e^{-\frac{l}{4c}}}\right)^2$

or, $\tan^2\frac{a}{4c}=\tanh^2\frac{l}{4c}$.

$\therefore$ $\tan\frac{a}{4c}=\tanh\frac{l}{4c}$. This is the equation which gives c.

Ex. 10. In a non-uniform string hanging under gravity the area of the cross section at any point is inversely proportional to the tension. Show that the curve is an arc of a parabola with its axis vertical. [C. H. 1959, 1963]

Take origin at the lowest point O and the tangent Ox there as the x-axis (see fig. of Ex. 4, p. 225).

If ω=wt. per unit length, T_0=tension at lowest point, ρ=radius of curvature at the point $P(x, y)$ where tangent makes an angle ψ with x-axis, we have $\omega=\dfrac{T_0}{\rho\cos^2\psi}$. [See Art. 6·10]

Given $\alpha=\dfrac{k}{T}$ where T is the tension and α the cross-section at P.

$\therefore$ $\omega=g\sigma\alpha=g\sigma\dfrac{k}{T}=\dfrac{g\sigma k\cos\psi}{T_0}$ [σ=density, $T\cos\psi=T_0$]

$$\therefore \quad \frac{T_0}{\rho \cos^2 \psi} = \frac{g\sigma k}{T_0} \cos \psi, \text{ or, } \rho = \lambda \sec^3 \psi \text{ where } \lambda = \text{constant}.$$

But, we know, $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \dfrac{\sec^3 \psi}{\dfrac{d^2y}{dx^2}}$

$$\therefore \quad \lambda \frac{d^2y}{dx^2} = 1 \qquad \text{or,} \quad \lambda \frac{dy}{dx} = x + A$$

At lowest point $x=0$, $\dfrac{dy}{dx} = \tan \psi = 0 \qquad \therefore \quad A = 0.$

$$\therefore \quad \lambda \frac{dy}{dx} = x \quad \therefore \quad \lambda y = \frac{x^2}{2} + B$$

At origin, when $x=0$, $y=0$. $\therefore B=0 \;\therefore\; \lambda y = \dfrac{x^2}{2}$ or, $x^2 = 2\lambda y$

a parabola whose axis viz. the y axis, is vertical.

Alit : We have $T_0 \dfrac{d}{ds}(\tan \psi) = \omega = \dfrac{g\sigma k}{T_0} \cos \psi$

$$\text{or,} \quad \frac{d}{ds}(\tan \psi) = \frac{1}{\lambda} \cos \psi = \frac{1}{\lambda} \cdot \frac{dx}{ds}$$

$$\therefore \quad \lambda \, . \, d(\tan \psi) = dx. \qquad \therefore \quad \lambda \frac{dy}{dx} = \lambda \tan \psi = x$$

(constant being zero)

$\therefore \quad \lambda y = \frac{1}{2}x^2$ (constant again zero) or, $x^2 = 2\lambda y$.

Ex. 11. If a string hangs in the form of an ellipse whose major axis is horizontal, prove that its weight per unit length at any point $\propto \dfrac{1}{b' y^2}$, where y is the depth of the point below the major axis and b' the length of the semi-conjugate diameter corresponding to the point.

Take origin at lowest point and the tangent at that point as x-axis ; if ψ be the angle the tangent at a point P makes with the x-axis, and ρ the radius of curvature at P, then we know, that the weight per unit length at P is $\omega = \dfrac{T_0}{\rho \cos^2 \psi}$. To find out ρ and $\cos^2 \psi$, we take the principal axes of the ellipse as our axes of co-ordinates. Let P be $(a \cos \theta, b \sin \theta)$.

Then $-\tan\psi=\frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}$ *or,* $\tan\psi=\frac{b\cos\theta}{a\sin\theta}$

$$\therefore\quad \sec^2\psi=1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}=\frac{b'^2}{a^2\sin^2\theta}=\frac{b^2b'^2}{a^2y^2}\quad\therefore\quad \cos^2\psi=\frac{a^2y^2}{b^2b'^2}.$$

We know, $\rho=\frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-x''y'}=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}=\frac{b'^3}{ab}.$

$$\therefore\quad \omega=\frac{T_0.\,ab.\,b^2b'^2}{b'^3.\,a^2y^2}=\frac{T_0b^3}{ab'y^2}\propto\frac{1}{b'y^2}.$$

Ex. 12. If a chain hangs in the form of the curve $y=-c\log_e\left(1-\frac{x^2}{c^2}\right)$, where c is *a* constant, express ω in terms of x.

[C. U. 1964]

When $x=0,\ y=0$ ∴ curve passes through the origin.

Differentiating, $\frac{dy}{dx}=\frac{2cx}{c^2-x^2}$. ∴ at origin when $x=0$,

$\tan\psi=0$ *or* $\psi=0$.

∴ x-axis is the tangent at the lowest point.

∴ we have, $T_0\tan\psi=\int_0^s\omega ds$ ∴ $T_0\frac{\sec^2\psi}{\rho}=\omega$

Now $\rho=\frac{\sec^3\psi}{\frac{d^2y}{dx^2}}$ ∴ $\omega=T_0\cos\psi.\frac{d^2y}{dx^2}$.

We have $\tan\psi=\frac{dy}{dx}=\frac{2cx}{c^2-x^2}$ ∴ $\frac{d^2y}{dx^2}=2c.\frac{c^2+x^2}{(c^2-x^2)^2}$

and $\cos\psi=\frac{c^2-x^2}{c^2+x^2}$,

$$\therefore\quad \omega=\frac{2cT_0}{c^2-x^2}\propto\frac{1}{c^2-x^2}.$$

MISCELLANEOUS EXAMPLES

1. A uniform chain hangs between two points A and B of which B is 10 ft. above A and no part of the chain hangs below B. If the tension at A is 80ω and makes an angle of 45° with the horizontal, show that the length of the chain is nearly 13·44 ft., ω being the weight of unit length of chain.

2. A uniform string ABC is fixed at A and passes over a smooth peg at B and the portion BC hangs vertically, the peg B being 10 ft. below the horizontal through A. The portion AB of the string is 30 ft. and BC is 25 ft. Show that the parameter of the catenary is $10\sqrt{6}$ ft.

3. If the tangents at the points P and Q of a catenary are at right angles, prove that the tension at the middle point of the arc PQ is equal to the weight of a length of the string equal to half the arc PQ.

4. A uniform string of length l and weight W is suspended between two points in the same horizontal line. If the sag in the middle is nl show that the tension at either end is $W.\frac{(4n^2+1)}{8n}$.

5. The ends of a uniform chain of length l are fastened to two light rings which can slide on two smooth rods in the same vertical plane, inclined in opposite ways at equal angles α to the vertical. Prove that the sag in the middle is $\frac{1}{2}l \tan\frac{\alpha}{2}$. If the rings be heavy and of equal weight and if the tension at the lowest point be equal to half the weight of the chain, show that the depths of the rings below the point of intersection of the rods is $\frac{l}{2}\cot\alpha\log(\sqrt{2}+1)$.

6. A heavy uniform string is suspended from two points A and B in the same horizontal line and to any point P of the string a heavy particle is attached. Prove that the two portions of the string are parts of equal catenaries. If θ, ϕ be the angles the tangents at P make with the horizon, α and β those made by the tangents at A and B, show that $\frac{\tan\theta+\tan\phi}{\tan\alpha+\tan\beta}$ is constant for all positions of P.

7. The ends of a uniform chain of length $2l$ are fastened to two weightless rings which can slide on a rough horizontal rod whose coefficient of friction is μ. At the middle point of the chain is attached a heavy particle whose weight is n times that of the chain. Show that the maximum distance between the rings is

$$2\mu l(n+1)\log\left[\frac{(n+1)(1+\sqrt{1+\mu^2})}{n+\sqrt{n^2+\mu^2(1+n)^2}}\right].$$

8. One end A of a heavy uniform string ABC of lengeh l is attached to a fixed point and the lower part BC of the string rests on a smooth inclined plane of inclination α; show that the directrix of the catenary formed is the horizontal line through the end C. If ψ be the angle the tangent at A makes with the horizontal prove that the length of BC is

$$\frac{l\cos\psi}{\cos\alpha\cos(\psi-\alpha)}.$$

9. A heavy uniform string passes over a fixed smooth peg; one end hangs freely and a part of the string of length $2l$ rests on a smooth inclined plane of inclination α. If the length of the part hanging freely be l, show that the whole length of the string is $4l\cos^2\alpha$ and that α is $\leq\frac{\pi}{6}$.

10. A heavy uniform string of length $4l$ passes through two small weightless smooth rings resting on a fixed horizontal bar. Prove that, if one of the rings be kept fixed and the other held at any point of the bar, the locus of the position of equilibrium of that end of the string which is further from the fixed ring may be represented by the equation

$$x = 2\sqrt{ly}.\ \log\frac{l}{y}.$$

11. A ring, of weight ωb, is attached to the middle point C of a string of length l and weight $l\omega$, which hangs symmetrically over two smooth pegs in the same horizontal line, the ends of the string hanging vertically. Show that the parameter c of the catenary is given by the equation $b + l = e^{\frac{a}{c}} [b + \sqrt{4c^2 + b^2}]$, where $2a$ is the distance between the pegs, and that the least value of l for which equilibrium is possible occurs when $\frac{1}{(c-a)^2} - \frac{1}{c^2} = \frac{4}{b^2}$.

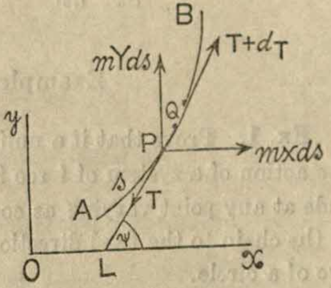
12. A string of length $2l$ hangs over two smooth pegs which are in the same horizontal line and at a distance $2a$ apart. The two ends of the string are free and its central portion hangs in a catenary. Show that in positions of equilibrium, the parameters of the catenaries are given by the roots of $ce^{\frac{a}{c}} = l$; and that there are two, one or no real values of c according as $l \gtreqless ae$. Show also that if c is the parameter of either of the two catenaries in which the string may hang when $l > ae$ the depth of the c. g. of the string below the pegs is $\frac{l^2 + c^2 - 2ac}{4l}$ and hence the catenary with the larger value of c is stable while the other is unstable.

13. A given length l of uniform heavy chain has one end fastened to a fixed point and hangs over a smooth peg in the same horizontal line at a distance $2a$ from it. Show that there are two positions of equilibrium or none according as $\frac{l}{a} \gtrless \sqrt{\frac{3}{1-\xi^2}}$, where ξ is the positive root of the equation $3e^{2\xi} = \frac{1+\xi}{1-\xi}$. Show also that if there are two positions of equilibrium then the one with the larger parameter of the catenary is stable.

14. A chain, of length $2l$ and weight $2W$, hangs with one end A attached to a fixed point in a smooth horizontal wire, and the other end B attached to a smooth ring which slides along the wire. Initially A and B are together. Show that the work done in drawing the ring along the wire till the chain at A is inclined at an angle of 45° to the vertical is $Wl[1 - \sqrt{2} + \log(1 + \sqrt{2})]$.

15. A bar, of length $2a$, has its ends fastened to those of a heavy string, of length $2l$, by which it is hung symmetrically over a peg. The weight of the bar is n times and the horizontal tension $\frac{1}{2}m$ times the weight of the string. Prove that $m^2 + n^2 = \left\{(n+1)\operatorname{cosech}\frac{a}{ml} - n\coth\frac{a}{ml}\right\}^2$.

6·11. To find the general Cartesian equations of equilibrium of a string under coplanar forces.



In the plane of the forces, take any point O as origin and Ox, Oy as rectangular axes of co-ordinates. In equilibrium, the string lies in the plane of the forces. Let AB be the string in equilibrium and let PQ be a small element of it. Let P be (x, y) and let its arcual distance measured from some fixed point on AB be s. Then length of PQ is ds. Let m be the mass of the string per unit length at P and let X, Y be the components of the external forces per unit mass acting at P. Consider the equilibrium of the element PQ. Forces acting on it are $mXds$, $mYds$ parallel to Ox, Oy, and the terminal tensions T at P and $T+dT$ at Q acting along the tangents at P and Q in the directions as shown in the figure.

Resolve the forces parallel to Ox. Resolved part of T is $T\dfrac{dx}{ds}$ in the direction xO. $T\dfrac{dx}{ds}$, the horizontal component of the tension is a function of s. At Q s has become $s+ds$; therefore at Q, the horizontal component of the tension is $\left(T\dfrac{dx}{ds}\right)+\dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)ds$ in the direction Ox. Hence we have, $\dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right)ds+mXds=0$

$$or,\quad \frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX=0\cdots\cdots(1)$$

Similarly resolving parallel to Oy, we have,

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY=0\cdots\cdots(2)$$

These are the Cartesian equations of equilibrium.

If m, X, Y be given, these equations determine the tension T at any point and also the form of the curve of equilibrium.

If the forces are not coplanar and X, Y, Z be their components parallel to the axes Ox, Oy, Oz, then in addition to the two equations (1) and (2), we have a third equation

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)+mZ=0.$$

Examples Worked

Ex. 1. Prove that if a uniform chain lie in one plane under the action of a system of force fixed in direction and of magnitude at any point varying as $\operatorname{cosec}^2\psi$, where ψ is the inclination of the chain to the fixed direction, the chain must lie along an arc of a circle.

Take a point O on the chain as origin and the line through O parallel to the fixed direction as the x-axis Ox. Then at every point P of the chain the force is parallel to Ox and its magnitude $\propto \operatorname{cosec}^2\psi$, when ψ is the angle the tangent at P makes with Ox. Hence, we have $X=k\operatorname{cosec}^2\psi$, $Y=0$ The general equations of equilibrium give,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+m.k.\operatorname{cosec}^2\psi=0 \quad\text{and}\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)=0$$

$$\therefore\quad T\sin\psi=\text{constant}=c. \qquad \frac{d}{ds}(c\cot\psi)+mk\operatorname{cosec}^2\psi=0$$

$$or,\quad -c\operatorname{cosec}^2\psi\frac{d\psi}{ds}+mk\operatorname{cosec}^2\psi=0$$

$\therefore\quad ds=\dfrac{c}{mk}d\psi=ad\psi$, where a is a constant.

Integrating $s=a\psi+A$.

If origin O be taken at the point of the chain where the tangent is parallel to the fixed direction then $s=0$ when $\psi=0$.

$\therefore\quad A=0\quad\therefore$ Equation of the form of the chain is $s=a\psi$, the well known intrinsic equation of a circle.

Ex. 2. The general equations of equilibrium may be used to find the form of equilibrium of a uniform string hanging under gravity. In this case, we have, $X=0$, $Y=-g$, $m=$ a constant. The general equations give,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+m.0=0 \quad\text{or}\quad T\frac{dx}{ds}=\text{const.}=T_0,\ \text{say}$$

or $T \cos \psi = T_0$ i.e. the horizontal tension is constant.

and $\dfrac{d}{ds}\left(T\dfrac{dy}{ds}\right) - mg = 0$

or $\dfrac{d}{ds}\left(T\dfrac{dy}{dx}\cdot\dfrac{dx}{ds}\right) = mg$ or $\dfrac{d}{ds}(T_0 \tan \psi) = mg$

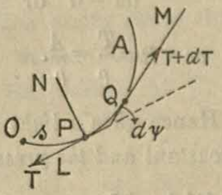
or $mg\,ds = T_0 d(\tan \psi)$ or $s = \dfrac{T_0}{mg} \tan \psi + B$

If we write $T_0 = cmg$, and if s is zero when $\psi = 0$

then $\mathbf{s = c \tan \psi}$.

This is the intrinsic equation of the common catenary.

6·12. *If a string is in equilibrium under the action of any forces, the two terminal tensions at the ends P and Q of a small element PQ of the string are equivalent to a force dT along the tangent at P and a force Tdψ along the inward normal at P, where T is the tension at P and ψ is the angle the tangent at P makes with a fixed line.*



For, resolving the two tensions T at P and $T+dT$ at Q, along the tangent LP and normal PN at P, we find that the sum of their resolved parts along the tangent LP

$= (T+dT) \cos d\psi - T$,

$d\psi$ being the small angle of first order between the tangents at P and $Q = dT$,

and the sum of their resolved parts along the inward normal

$PN = (T+dT) \sin d\psi = T d\psi$,

for $\cos d\psi = 1 - \dfrac{(d\psi)^2}{2}\eta$, where $|\eta| < 1, = 1$

and $\sin d\psi = (d\psi) - \dfrac{(d\psi)^3}{6}\eta'$, where $|\eta'| < 1, = d\psi$,

small quantities of higher order than the first being neglected.

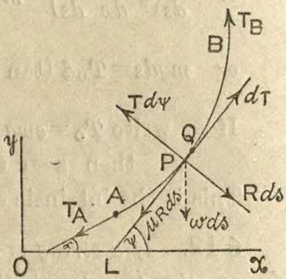
Thus the two terminal tensions are equivalent to a force dT along the tangent (in the direction in which s increases) and a force Tdψ along the inward normal.

This result is very important; in problems on strings it is very convenient to replace the terminal tensions by these two forces.

6·13. Constrained string—Light string, smooth surface.

A light inextensible string rests in equilibrium in one plane on a smooth surface under forces acting at its ends.

Consider the equilibrium of a small element PQ of the string; let s be the arcual distance of P measured from some fixed point on the string then $PQ=ds$. Let T be the tension at P, ψ the angle the tangent PL at P makes with Ox, and R the normal reaction of the surface per unit length of the string at P. Then the forces acting on the element PQ are the terminal tensions which are equivalent to a force dT along the tangent LP and a force $Td\psi$ along the inward normal at P and the reaction Rds along the outward normal at P. Resolving along and perpendicular to the tangent at P, we have,



$$dT=0 \quad \text{or} \quad T=\text{const.}=A, \quad \text{and} \quad Rds=Td\psi$$

$$\text{or} \quad R=\frac{T}{\rho}=\frac{A}{\rho}.$$

Hence *when a light string rests on a smooth surface, the tension is constant and the pressure at any point varies as the curvature.*

6·14. Heavy string, smooth surface.

A heavy uniform string rests in equilibrium in one plane on a smooth surface under forces acting at its ends.

Consider the equilibrium of a small element PQ of the string of length ds (fig. Art 6·13). Let T be the tension at P along the tangent PL which makes an angle ψ with a horizontal line Ox. Let w be the wt. and R the normal reaction, both per unit length of the string at P. Then the element PQ is in equilibrium under the forces dT along the tangent, $Td\psi$ along the inward normal, Rds along the outward normal, (the string is assumed to be pressed against the lower side of the surface) and the weight wds along the vertical. Resolving along and perpendicular to the tangent at P, we have

$$dT=wds\sin\psi=wds\frac{dy}{ds}=wdy \quad \therefore \quad T=wy+A \quad (1)$$

and $Rds+wds\cos\psi=Td\psi$ or $\frac{T}{\rho}=R+w\cos\psi$...(2)

If T_1, T_2 be the tensions at two points whose ordinates are y_1, y_2, we have $T_2-T_1=w(y_2-y_1)$.

Hence if a heavy uniform string rests on a smooth surface *the difference of the tensions at any two points is equal to the weight of a portion of the string whose length is equal to the height of one point above the other.* It is an important result. The second equation gives the pressure at any point, T being known by equation (1).

6·15. Light string, rough surface.

A light inextensible string rests in limiting equilibrium in one plane over a rough surface under forces acting at its ends.

As in the previous articles we consider the equilibrium of an element PQ of the string of length ds (fig. Art. 6·13); we assume that the string is about to move in the direction AB. The force of friction is μRds along PL. The string being light there is no weight ωds. The other forces remain the same as before. Resolving along the tangent and the normal at P, we have, $dT=\mu Rds$ and $Td\psi=Rds$.

Eliminating R, $\frac{dT}{T}=\mu d\psi$ *or*, $T=Ae^{\mu\psi}$...(1) ; and $R=\frac{T}{\rho}$...(2)

If T_1, T_2 be the tensions at two points where the tangents make angles ψ_1, ψ_2 with Ox, we have, $T_2=T_1e^{\mu(\psi_2-\psi_1)}$.

Hence if *a light string is in limiting equilibrium on a rough surface and A, B are two points of the string and the string tends to move in the direction A to B, then* $T_B=T_Ae^{\mu\theta}$, *where* θ *is the angle between the tangents at A and B.* To determine the angle θ, first let the tangents at A and B both have the direction of AB and then imagine that the tangent at A rolls over the string till it coincides with the tangent at B; the angle turned round is θ.

The pressure at any point is given by (2), $R=\frac{T}{\rho}$.

6·16. Heavy string, rough surface.

This is the most general case.

A uniform heavy string rests in equilibrium in one plane on a rough surface under forces acting at its ends.

As before we consider the equilibrium of a small element PQ of the string of length ds (fig. Art. 6·13). The forces acting on the element are the same as those in the previous article with an additional vertical force ωds. Resolving along the tangent and the normal at P, we have, $dT-\omega ds\sin\psi-\mu Rds=0$······(1)

$$\text{and}\quad Td\psi-\omega ds\cos\psi-Rds=0\cdots\cdots(2)$$

$$\text{Eliminating } R,\quad \frac{dT}{d\psi}-\mu T=\rho\omega(\sin\psi-\mu\cos\psi),$$

Multiplying by the integrating factor $e^{-\mu\psi}$ and integrating we have, $Te^{-\mu\psi}=\int\rho\omega e^{-\mu\psi}(\sin\psi-\mu\cos\psi)d\psi+A.$

From the equation of the curve in which the string rests, we can find ρ in terms of ψ. Putting the value of ρ and integrating we have, $Te^{-\mu\psi}=f(\psi)+A.$

This equation gives T.

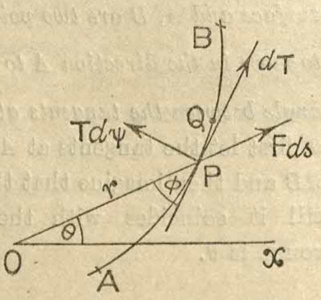
From (2), we have, $\dfrac{T}{\rho}=\omega\cos\psi+R$ *or*, $R=\dfrac{T}{\rho}-\omega\cos\psi.$

T being known, this gives the pressure R.

6·17. Central forces.

An inextensible string of given length is attached to two fixed points and is in equilibrium in one plane under the action of a given central force ; to find the form of the curve of equilibrium.

Let AB be the string in equilibrium where A and B are the fixed points ; and let O be the centre of force. Let P be any point of the string whose polar co-ordinates are (r, θ) referred to O as pole and let ϕ be the angle between OP and the tangent at P. Let s be the arcual distance of P measured from some fixed point A on the curve. Consider the equilibrium of a small element PQ of the string of length ds. If F be



the central force per unit length of the string, considered positive when repulsive, the central force acting on element PQ is Fds. The other forces acting on PQ are the two terminal tensions which are equivalent to a force dT along the tangent at P and a force $Td\psi$ along the inward normal at P. Resolving the forces along the tangent and the normal at P, we have,

$$dT+Fds\cos\phi=0\ldots\ldots(1)$$

and $Td\psi=Fds\sin\phi\ldots\ldots\ldots(2)$

From (1), we have, $dT=-F.ds\cdot\dfrac{dr}{ds}=-Fdr.$

$$\therefore\quad T=A-\int Fdr\ldots\ldots\ldots\ldots(3)$$

Since F is given as a function of r, this equation gives T.

From (2), we have, $T=F\cdot\rho\cdot\dfrac{p}{r}$, where p is the perpendicular from O on the tangent at P and $\sin\phi=\dfrac{p}{r}$

$$=F.r\frac{dr}{dp}\cdot\frac{p}{r}=-dT\cdot\frac{p}{dp}$$

$or,\ \ Tdp+pdT=0\ \ or,\ \ d(Tp)=0\ \ or,\ \ Tp=\text{constant}=B\ldots(4)$

T being known from (3) in terms of r, (4) gives the (p, r) equation of the curve of equilibrium. To find the polar equation we use the relation $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. The solution of the differential equation in (r, θ) will contain three arbitrary constants. The three constants are determined by the fact that the string has two given points A, B on it and that the length of the string AB is given. This completely determines the form of the curve.

Secondly suppose the form of the string is given and we have to find the law of force.

First we find the (p, r) equation of the curve. Then using (4) we find T in terms of r. Finally F is given by the equation $F=-\dfrac{dT}{dr}$. Thus the central force is found.

Examples Worked

Ex. 1. A single movable pulley, of weight W, is just supported by a power P which is applied to one end of a light cord which goes under the pulley and is then fastened to a fixed point; show that, if θ be the angle subtended at the centre by the part of the string in contact with the pulley, then

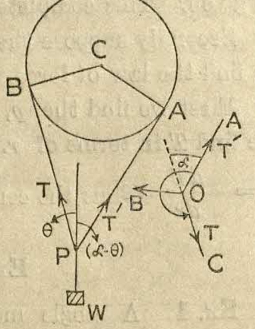
$$P^2(1-2e^{\mu\theta}\cos\theta+e^{2\mu\theta})=W^2$$

Let $ABCD$ be the string, A being the fixed end, BC the portion in contact with the pulley and D the end where power P is applied. As the pulley is just supported it tends to move down in the direction B to C. So relatively the string tends to move in the direction C to B. $\therefore$ $T_B > T_C$ $\therefore$ $T_B = T_C . e^{\mu\theta}$, for angle between the tangents at B, C both in direction CB, is equal to the angle between the normals at B, C, which is θ. But $T_C = P$ $\therefore$ $T_B = Pe^{\mu\theta}$. The forces which keep the pulley in equilibrium are T_C in the direction CD and T_B in the direction BA and the angle between CD and BA is $(\pi-\theta)$. Hence the resultant of the forces P and $Pe^{\mu\theta}$ acting at an angle $(\pi-\theta)$ is W.

$$\therefore \quad W^2 = P^2 + P^2e^{2\mu\theta} + 2P.Pe^{\mu\theta}\cos(\pi-\theta)$$
$$= P^2(1-2e^{\mu\theta}\cos\theta+e^{2\mu\theta})$$

Ex. 2. A heavy particle is attached to an endless light inextensible string which passes over a rough pulley fixed in a vertical plane. If the straight parts of the string are inclined to each other at an angle α, prove that, for limiting equilibrium, their inclinations to the vertical are $\tan^{-1}\dfrac{\sin\alpha}{\cos\alpha+e^{\pm\mu(\alpha+\pi)}}$.

C is the centre of the fixed pulley; wt. W is suspended from point P of the string and PA, PB are the straight parts of the string which touch pulley at A and B. Let PB make an angle θ with vertical then PA makes the angle $(\alpha-\theta)$. Let T, T' be the tensions in PA and PB. For equilibrium of W, we have, by Lami's theorem,



$$\frac{T}{\sin(\alpha-\theta)}=\frac{T'}{\sin\theta}\cdots\cdots(1).$$

Now, we have to find the relation between T and T'. As shown in the figure, P tends to move down. ∴ The string at B tends to move in the direction BP. ∴ $T>T'$; and the direction of T' is to be rotated through an angle $(\alpha+\pi)$ to coincide with the direction of T *i. e.* BP. ∴ $T=T'e^{\mu(\alpha+\pi)}$.

$$\therefore \text{ From (1), we have } \sin(\alpha-\theta)=\sin\theta . e^{\mu(\alpha+\pi)} \quad \ldots\ldots(2)$$

$$or, \quad \sin\alpha\cot\theta-\cos\alpha=e^{\mu(\alpha+\pi)}$$

$$or, \quad \sin\alpha\cot\theta=\cos\alpha+e^{\mu(\alpha+\pi)}$$

$$\therefore \quad \tan\theta=\frac{\sin\alpha}{\cos\alpha+e^{\mu(\alpha+\pi)}}.$$

In (2) write $\alpha-\theta=\phi$, then, we have $\sin\phi . e^{-\mu(\alpha+\pi)}=\sin(\alpha-\phi)$

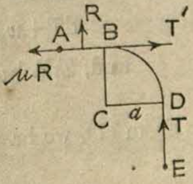
$$or, \quad e^{-\mu(\alpha+\pi)}=\sin\alpha\cot\phi-\cos\alpha.$$

$$\therefore \quad \tan\phi=\frac{\sin\alpha}{\cos\alpha+e^{-\mu(\alpha+\pi)}}.$$

Hence the inclinations of the strings are

$$\tan^{-1}\frac{\sin\alpha}{\cos\alpha+e^{\pm\mu(\alpha+\pi)}}.$$

Ex. 3. A heavy chain, of length l, rests partly on a rough table, and the remainder after passing over the smooth edge of the table, which is rounded off in the form of a cylinder of radius a, hangs freely down. If the coefficient of friction is μ, show that the least length on the table is $\dfrac{1}{\mu+1}\left[l-\dfrac{a\pi}{2}+a\right]$.



Let $ABDE$ be the chain; AB rests on the rough table and DE hangs freely. Let $AB=x$, $DE=y$, ω=wt. per unit length of the string, and let F be the friction on portion AB. Let T' be the tension of the chain at B and T that at D. Since the rounded part is smooth, ∴ $T'=T+\omega . a$, a being the height of B above D. Again, for equilibrium of the portions AB and DE, we have, $T'=F$, $R=\omega x$; $T=\omega y$, and also we have $x+y+a\frac{\pi}{2}=l$.

$$\therefore \quad T' = \omega(a+y) = \omega\left(a+l-\frac{a\pi}{2}-x\right) \leqslant \mu R \leqslant \mu\omega x$$

$$\therefore \quad \left(a+l-\frac{a\pi}{2}\right) \leqslant x(\mu+1)$$

$$\therefore \quad \text{Least value of } x = \frac{1}{\mu+1}\left[a+l-\frac{a\pi}{2}\right].$$

Ex. 4. A uniform heavy string rests on a smooth parabola, whose axis is vertical and vertex upwards, so that its ends are at the extremities of the latus rectum. Show that the pressure on the curve, at the point where the tangent makes an angle ϕ with the horizontal, is $\frac{w}{2}(2\cos^3\phi + \cos\phi)$, where w is the weight per unit length of the string. [C. H. 1953]

Show that the pressure at the vertex is $\frac{3w}{2}$.

[C. H. 1958, 1960]

We take the origin at O, the vertex, and the tangent at O as the axis of x. The equation of the parabola is $x^2 = 4ay$......(1) where $4a$ = latus rectum (use fig. of Ex. 5 below replacing ψ by ϕ and omitting μRds). Consider the equilibrium of a small element of the string of length ds at P, where the tangent makes an angle ϕ with Ox. Forces acting on the element are (1) its weight wds acting vertically downwards, (2) the normal reaction of the curve Rds, acting along the normal outwards, (3) the two terminal tensions, which are equivalent to a force dT along the tangent and a force $Td\phi$ along the inward normal. Resolving the forces along and perpendicular to the tangent we get,

$$dT + w\,ds\sin\phi = 0\text{.........(1)}$$

$$\text{and, } Td\phi + wds\cos\phi = Rds\text{.........(2)}$$

From (1), we have, $dT = -wds.\frac{dy}{ds} = -w.dy$

$$\therefore \quad T = A - wy.$$

At the free end of the string at the end of the latus rectum *i.e.* at $(2a, a)$, $T = 0$.

$$\therefore \quad T = w(a-y)\text{.........(3)}.$$

From (2), we have, $R=\frac{T}{\rho}+w\cos\phi$

∴ $R=\frac{w(a-y)}{\rho}+w\cos\phi$........(4)

We have to express y and ρ in terms of ϕ.

Differentiating (1), $2x=4a\frac{dy}{dx}$ ∴ $\tan\phi=\frac{dy}{dx}=\frac{x}{2a}$.

∴ $y=\frac{x^2}{4a}=a\tan^2\phi$. Also $\frac{d^2y}{dx^2}=\frac{1}{2a}$

$$\therefore\quad \rho=\frac{(1+\tan^2\phi)^{\frac{3}{2}}}{\frac{1}{2a}}=2a\sec^3\phi.$$

$$\therefore\quad \text{From (4), } R=\omega\left[\frac{(a-a\tan^2\phi)}{2a}\cos^3\phi+\cos\phi\right]$$

$$=\frac{\omega}{2}[\cos^3\phi-\sin^2\phi\cos\phi+2\cos\phi]$$

$$=\frac{\omega}{2}[\cos^3\phi+\cos\phi.\cos^2\phi+\cos\phi]$$

$$=\frac{\omega}{2}[2\cos^3\phi+\cos\phi].$$

If the pressure at the vertex be required, then in (4) we have to put $y=0$, $\phi=0$ and $\rho=$radius of curvature at the vertex, which is well-known to be 2a.

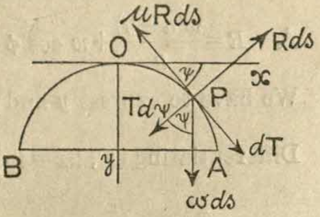
∴ Pressure at vertex$=\frac{\omega a}{2a}+\omega=\frac{3\omega}{2}$.

Ex. 5. A uniform heavy chain rests on a rough cycloid, whose axis is vertical and vertex upwards, one end of the chain being at the vertex and the other at a cusp ; if the equilibrium be limiting, show that $(1+\mu^2)e^{\frac{\mu\pi}{2}}=3$.

We take the origin at the vertex O and the tangent Ox at O as the x-axis. The equation of the cycloid is

$$\left.\begin{aligned} x&=a(\theta+\sin\theta)\\ y&=a(1-\cos\theta)\end{aligned}\right\}\text{.....(1)}$$

Let the tangent at $P(\theta)$ make an angle ψ with Ox. We know, $\psi=\frac{\theta}{2}$. The forces acting on a small element of the chain of length ds at P are (1) its weight wds, acting vertically downwards, (2) the normal reaction of the curve Rds, acting along the outward normal, (3) the friction μRds acting along the tangent and (4) the terminal tensions which are equivalent to a force dT along the tangent and a force $Td\psi$ along inward normal. Resolving the forces along and perpendicular to the tangent, we get,



$$dT+wds\sin\psi-\mu Rds=0\ldots\ldots(1)$$

$$\text{and } Td\psi+wds\cos\psi-Rds=0\ldots\ldots(2).$$

Eliminating R, we get,

$$dT-\mu Td\psi=wds(\mu\cos\psi-\sin\psi)$$

$$=w.4a\cos\psi(\mu\cos\psi-\sin\psi)d\psi$$

$$[\because\ s=4a\sin\psi]$$

$$\therefore\ \frac{dT}{d\psi}-\mu T=4aw\sec\lambda.\cos\psi\sin(\lambda-\psi)\quad(\text{where } \mu=\tan\lambda)$$

$$or,\ \frac{d}{d\psi}\left[Te^{-\mu\psi}\right]=e^{-\mu\psi}.2aw\sec\lambda.[\sin\lambda-\sin(2\psi-\lambda)]$$

$$\therefore\ \left[Te^{-\mu\psi}\right]_{\substack{T=0\\ \psi=0}}^{\substack{T=0\\ \psi=\frac{\pi}{2}}}=2aw\sec\lambda\int_{\psi=0}^{\frac{\pi}{2}}e^{-\mu\psi}[\sin\lambda-\sin(2\psi-\lambda)]d\psi$$

$$\left[\because\ T=0 \text{ when } \psi=0 \text{ and when } \psi=\frac{\pi}{2}\right]$$

$$\therefore\ 0=\left[\frac{e^{-\mu\psi}\sin\lambda}{-\mu}\right]_{\psi=0}^{\frac{\pi}{2}}$$

$$-\left[\frac{e^{-\mu\psi}}{\mu^2+4}\{-\mu\sin(2\psi-\lambda)-2\cos(2\psi-\lambda)\}\right]_0^{\frac{\pi}{2}}.$$

or, $$\frac{\sin\lambda}{\mu}\left(1-e^{-\frac{\mu\pi}{2}}\right)\left(\mu^2+4\right)$$

$$=e^{-\frac{\mu\pi}{2}}(-\mu\sin\lambda+2\cos\lambda)-(\mu\sin\lambda+2\cos\lambda)$$

or, $$\cos\lambda\,(e^{\frac{\mu\pi}{2}}-1)(\mu^2+4)$$

$$=(-\mu\sin\lambda+2\cos\lambda)-e^{\frac{\mu\pi}{2}}(\mu\sin\lambda-2\cos\lambda)$$

or, $$(e^{\frac{\mu\pi}{2}}-1)(\mu^2+4)=(-\mu^2+2)-e^{\frac{\mu\pi}{2}}(\mu^2-2)$$

or, $e^{\frac{\mu\pi}{2}}(2\mu^2+2)=6$ or, $(\mu^2+1)e^{\frac{\mu\pi}{2}}=3.$

Ex. 6. A uniform inextensible string, of length l, hangs in limiting equilibrium over a fixed rough cylinder of radius a whose axis is horizontal. Show that the length of the greater of the two vertical portions is $\dfrac{l-a\pi}{l+e^{-\mu\pi}}+\dfrac{2\mu a}{1+\mu^2}$.

Take the highest point O of the vertical circle in which the string rests as origin and the tangent at O as x-axis (use fig. of Ex. 5 above). Consider the quilibrium of the small element of the string at P which makes an angle ψ with Ox. Then $s=$arc $OP=a\psi$ $\therefore$ length of element $ds=ad\psi$. Forces acting on the element are its weight wds vertically down, normal reaction Rds along outward normal, friction μRds along tangent, opposing the motion of string and the terminal tensions which are equivalent to forces dT along tangent and $Td\psi$ along inward normal. We assume P to be on the side of the greater portion of the string. Resolving along the tangent and the normal, we have,

$$dT+wds\sin\psi-\mu Rds=0$$

$$\text{and } Td\psi+wds\cos\psi-Rds=0$$

Eliminating R, we have, $dT-\mu d\psi=wds(\mu\cos\psi-\sin\psi)$

or, $\dfrac{dT}{d\psi}-\mu=-wa\sec\lambda\sin(\psi-\lambda)$, where $\mu=\tan\lambda$

Multiplying by the integrating factor $e^{-\mu\psi}$ and integrating between $\psi_1=-\frac{\pi}{2}$ and $\psi_2=\frac{\pi}{2}$ where $T_1=wy$, and $T_2=wx$, where x is the greater one of the hanging portions and y the smaller one we have,

$$\int_{\psi=\frac{\pi}{2},\, T=wy}^{\psi=\frac{\pi}{2},\, T=wx} d\left(Te^{-\mu\psi}\right)=-aw\sec\lambda\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\mu\psi}\sin(\psi-\lambda)d\psi,$$

$$\text{or,}\quad \left(wxe^{-\frac{\mu\pi}{\psi}}-wye^{\frac{\mu\pi}{2}}\right)$$

$$=\frac{-aw\sec\lambda}{\mu^2+1}\cdot\left[e^{-\mu\psi}(-\mu\sin\overline{\psi-\lambda}-\cos\overline{\psi-\lambda})\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

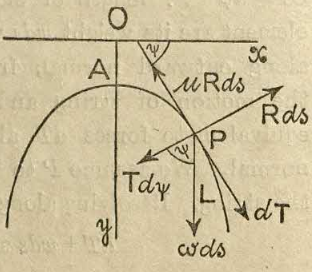
$$=-aw\cos\lambda\cdot\left[e^{-\frac{\mu\pi}{2}}(-2\sin\lambda)-e^{\frac{\mu\pi}{2}}.2\sin\lambda\right]$$

$$\text{or,}\quad xe^{-\mu\pi}-y=a\sin 2\lambda(e^{-\mu\pi}+1)$$

$$\text{or,}\quad x(e^{-\mu\pi}+1)=(x+y)+a\sin 2\lambda(e^{-\mu\pi}+1)$$

$$\text{or,}\quad x=\frac{l-a\pi}{1+e^{-\mu\pi}}+\frac{a.2\mu}{1+\mu^2}.\qquad [\because\ x+y+a\pi=l\,]$$

Ex. 7. If a heavy string be laid on a rough catenary, with its vertex upwards and axis vertical, so that one extremity is at the vertex, the string will just rest if its length be equal to the parameter of the catenary, provided the coefficient of friction is $\frac{2\log 2}{\pi}$. [C. H. 1962]



We take the directrix and the axis of the catenary as axes of x and y. Considering the forces which act on an element of length ds at P and resolving the forces along and perpendicular to the tangent, we get, as in example above,

$$dT+wds\sin\psi-\mu Rds=0$$

$$\text{and}\quad Td\psi+wds\cos\psi-Rds=0$$

Eliminating R, $dT-\mu Td\psi=wds(\mu\cos\psi-\sin\psi)$

$$[\, s=c\tan\psi \quad \therefore \quad ds=c\sec^2\psi d\psi.\,]$$

$$\therefore \quad \frac{dT}{d\psi}-\mu T=w.c\sec^2\psi(\mu\cos\psi-\sin\psi)$$

$$\therefore \quad \frac{d}{d\psi}[Te^{-\mu\psi}]=cwe^{-\mu\psi}.[\mu\sec\psi-\sec\psi\tan\psi]$$

$$=cw\frac{d}{d\psi}[-e^{-\mu\psi}.\sec\psi].$$

Integrating, $Te^{-\mu\psi}=cw[A-e^{-\mu\psi}.\sec\psi]$

or, $T=cw[Ae^{\mu\psi}-\sec\psi]$.

When $\psi=0$, $T=0$ $\therefore$ $A=1$ $\therefore$ $T=cw[e^{\mu\psi}-\sec\psi]$.

If length of the string be c, the other end of the string will be at Q where ψ is given by $c=c\tan\psi$ or, $\psi=\frac{\pi}{4}$ and therefore, when $\psi=\frac{\pi}{4}$, we must have $T=0$

$$\therefore \quad e^{\frac{\mu\pi}{4}}-\sec\frac{\pi}{4}=0 \quad or, \quad e^{\frac{\mu\pi}{4}}=\sqrt{2} \quad or, \quad \text{squaring } e^{\frac{\mu\pi}{2}}=2$$

$$\therefore \quad \frac{\mu\pi}{2}=\log 2 \quad or, \quad \mu=\frac{2\log 2}{\pi}.$$

Hence if μ be $\frac{(2\log 2)}{\pi}$, the string of length equal to c will rest in limiting equilibrium on the catenary with one end at the vertex.

Ex. 8. Find the law of force in the case of strings resting in the form of the following curves under a central force F from the pole :

(1) Parabola, focus the pole

(2) Rectangular hyperbola, centre the pole

(3) $r^n\cos n\theta=a^n$.

(1) The (p, r) equation of a parabola referred to focus as pole is $p^2=ar$

or, $p=a^{\frac{1}{2}}.r^{\frac{1}{2}}$. We have $pT=c$, a constant.

$\therefore\ T=\dfrac{c}{a^{\frac{1}{2}}}\cdot r^{-\frac{1}{2}}.\quad \therefore\ F=-\dfrac{dT}{dr}=\dfrac{c}{a^{\frac{1}{2}}}\cdot\tfrac{1}{2}r^{-\frac{3}{2}}\propto r^{-\frac{3}{2}}$; as F is $+ve$, it is a respulsive force.

(2) The (p, r) equation of a rect. hyperbola is $pr=a^2$

$or,\ p=\dfrac{a^2}{r}.\quad \therefore\ T\cdot\dfrac{a^2}{r}=c\quad or,\ T=\dfrac{c}{a^2}r.\quad \therefore\ F=-\dfrac{dT}{dr}=-\dfrac{c}{a^2}.$

$\therefore$ F is constant and attractive.

(3) Taking log. of both sides, we have,

$$n\log r+\log\cos n\theta=n\log a$$

Differentiating with respect to r, we have,

$$\frac{n}{r}-n\tan n\theta\frac{d\theta}{dr}=0.$$

$$\therefore\ \tan\phi=r\frac{d\theta}{dr}=\cot n\theta\quad or,\ \phi=\frac{\pi}{2}-n\theta$$

$$\therefore\ p=r\cos n\theta=\frac{a^n}{r^{n-1}}\quad \therefore\ T\cdot\frac{a^n}{r^{n-1}}=c\quad or,\ T=\frac{c}{a^n}\cdot r^{n-1}$$

$$\therefore\ F=-\frac{dT}{dr}=-\frac{c}{a^n}(n-1)r^{n-2}\propto r^{n-2}$$

If $n>1$, sign of F is negative.

$\therefore$ F is attractive and if $n<1$, F is repulsive

Ex. 9. If the curve of equilibrium is an ellipse whose focus is the centre of force, the force at any point $\propto\dfrac{1}{rb'}$, where b' is the semi-conjugate diameter corresponding to the point and r the focal distance of the point.

The (p, r) eqn. of the ellipse referred to focus is $\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1.$

Since $Tp=A\ \therefore\ \dfrac{T^2b^2}{A^2}=\dfrac{2a}{r}-1.$ Differentiating w.r to r, we have $\dfrac{2Tb^2}{A^2}\cdot\dfrac{dT}{dr}=-\dfrac{2a}{r^2}\quad\therefore\ F=-\dfrac{dT}{dr}=\dfrac{aA^2}{b^2Tr^2}=\dfrac{aAp}{b^2r^2}.$ Now, from co-ordinate geometry, we know, $\dfrac{p}{r}=\dfrac{b}{b'}\quad\therefore\ F=\dfrac{aA}{b^2}\cdot\dfrac{b}{b'r}\propto\dfrac{1}{rb'}.$

Ex. 10. A string rests in the form of a plane curve under the action of a central repulsive force ; if the force at any point be proportional to the curvature, prove that the curve is a parabola.

Considering the equilibrium of a small element of length ds at $P(r, \theta)$, and resolving perpendicular to the tangent at P, if Fds be the force acting on the element, we have, $Td\psi = Fds \sin\phi$.

or, $T = F.\rho \sin\phi$.........(1)

Also we know $Tp = c$, a constant.........(2).

Since $F \propto \rho$, let $F = \frac{k}{\rho}$ $\therefore$ From (1), $T = \frac{k}{\rho}\rho.\frac{p}{r}$

$$[\because \; p = r\sin\phi\;] = \frac{kp}{r}.$$

$\therefore$ From (2), $\frac{kp}{r}.p = c$ *or*, $p^2 = \frac{c}{k}r = ar$.

This is the well-known (p, r) eqn. of a parabola referred to its focus as pole. Hence the curve is a parabola and the centre of force is at its focus.

Alit. $-\frac{dT}{dr} = F = \frac{k}{\rho} = \frac{k}{r}\frac{dp}{dr}$ $\therefore$ $rdT + kdp = 0$ Now $T = \frac{c}{p}$

$\therefore$ $dT = -\frac{c}{p^2}dp$. $\therefore$ $-\frac{cr}{p^2} + k = 0$ *or*, $p^2 = \frac{c}{k}r = ar$.

Ex. 11. A string of infinite length has one extremity attached to a fixed point A, and passing through a small smooth fixed ring at B stretches to infinity in a straight line, the whole being under the influence of a central repulsive force $= \mu.u^n$, where $n > 1$. Show that the form of the string between A and B is $r^{n-2} = b^{n-2}\cos(n-2)\theta$.

Let O be the centre of force; take O as origin, and OB as x-axis, and let $OB = a$. The string evidently stretches along OB. At P on Ox where $OP = x$, consider equilibrium of the element PQ of length dx. The terminal tensions at P, Q are T and $T + dT$ in the directions xO and Ox and the repulsive force is $\frac{\mu}{x^n}dx$ along Ox.

$\therefore$ $dT + \frac{\mu}{x^n}dx = 0$ $\therefore$ $T = \frac{\mu}{(n-1)}.\frac{1}{x^{n-1}} + A$.

As $T = 0$ when $x = \infty$, $\therefore$ $A = 0$ (as $n > 1$)

$$\therefore \quad T=\frac{\mu}{(n-1)x^{n-1}}, \quad \therefore \quad \text{Tension at } B \text{ is } T_B=\frac{\mu}{(n-1)a^{n-1}} \qquad \ldots\ldots\ldots(1)$$

Now, for the curved portion of the string, at a point (r, θ), we have, $\frac{dT}{dr}=-\frac{\mu}{r^n}$ $\therefore$ $T=\frac{\mu}{(n-1)r^{n-1}}$, no constant is added as when $r=a$, $T=T_B$. Also we have $pT=c$. $\therefore$ $\frac{c}{p}=T=\frac{\mu}{(n-1)r^{n-1}}$;

Squaring and substituting the value of $\frac{1}{p^2}$, we have,

$$c^2\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]=\frac{\mu^2}{(n-1)^2 r^{2n-2}}$$

$$or, \quad \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{k^2}{r^{2n-2}} \quad or, \quad \frac{dr}{d\theta}=\pm\frac{\sqrt{k^2-r^{2n-4}}}{r^{n-3}},$$

where $k=$ const.

$$\therefore \quad \pm\theta+\text{constant}=\int\frac{r^{n-3}\,dr}{\sqrt{k^2-r^{2n-4}}};$$

to integrate put $r^{n-2}=k\cos\phi$

$$=\int\frac{-1}{n-2}d\phi$$

$$or, \quad \phi=\pm(n-2)(\theta+D), \; D=\text{a constant.}$$

$\therefore$ $r^{n-2}=k\cos\{(n-2)(\theta+D)\}$. When $\theta=0$, $r=a$,

$\therefore$ $a^{n-2}=k\cos(n-2)D$. This determines D, say $D=\alpha$.

$\therefore$ $r^{n-2}=k\cos[(n-2)(\theta+\alpha)]$. Take Ox' as the new initial line where $x'Ox=\alpha$. The equation becomes $r^{n-2}=k\cos(n-2)\theta$. Writing b^{n-2} for k, the equation is $r^{n-2}=b^{n-2}\cos(n-2)\theta$.

This is the equation of the curve of equilibrium of the string between A and B.

EXAMPLES

1. Two weights P, Q hang in limiting equilibrium from a light string which passes over a rough circular cylinder in a plane perpendicular to the axis, which is horizontal. If P be on the point of descending, show that the weight which may be added to Q without causing it to descend is $\frac{(P^2-Q^2)}{Q}$.

2. Two equal weights, each equal to P, are attached to the two ends of a light string which hangs over a rough vertical circle of coefficient of friction μ. Show that either weight is to be increased by $P(e^{\mu\pi}-1)$ in order that it may descend.

3. A circle has its plane vertical, and is pressed against a vertical wall by a string fixed to a point in the wall above the circle. The string sustains a weight P, the coefficient of friction between the string and circle is μ, and the wall is perfectly rough. When the circle is on the point of sliding, prove that, if W be the weight of the circle and θ the angle between the string and the wall, $P(1+\cos\theta)e^{\mu\theta}=W+2P$.

4. A light string passes over a rough fixed horizontal cylinder and is attached to a weight W; P is the weight which will just raise W, and P' the weight which will just sustain W; show that if R, R' are the corresponding resultant pressures of the string on the cylinder, $P : P' = R^2 : R'^2$.

5. A light string passes over two rough pegs A and B in the same horizontal line at a distance $2a$ apart. The ends are attached to a weight C and in the position of limiting equilibrium the angle ACB is a right angle. Show that the horizontal distance of C from the middle point of AB is $a\tanh\frac{3\mu\pi}{2}$, where μ is the coefficient of friction.

6. A, B, C are three rough pegs in a vertical plane; P, Q, R are the greatest weights which can be severally supported by a weight W when connected with it by strings passing over A, B, C, over A, B, and over B, C respectively. Show that the coefficient of friction at B is $\frac{1}{\pi}\log\frac{QR}{PW}$.

7. Four rough pegs are at the angular points of a square in a vertical plane with its sides horizontal and vertical. Over each peg passes a string supporting a weight W, and the other ends of these four strings are knotted together. Show that the greatest weight that can be attached to this knot, so that it may remain in equilibrium at the centre of the square, is $2\sqrt{2}We^{\frac{\mu\pi}{4}}\sinh\frac{\pi\mu}{2}$.

8. A string whose length is l, is hung over two rough pegs at a distance a apart in a horizontal line. If one free end of the string is as much as possible lower than the other, the inclination of the tangent to the vertical at either peg is given by the equation $\frac{l}{a}\sin\theta\log\cot\frac{\theta}{2}=\cos\theta+\cosh\mu(\pi-\theta)$. Show also that the ratio of the vertical portions is $e^{2\mu\pi} : e^{2\mu\theta}$ and the length of the portion between the pegs is $a\cot\theta\div\log\cot\frac{\theta}{2}$.

9. A uniform string rests on the upper half of a smooth vertical circle, its ends reaching to the horizontal diameter. Prove that the pressure at the highest point is 2ω, where ω is the weight per unit length of the string.

10. A uniform string hangs beneath the lower half of a smooth vertical circle. Show that if the pressure at the lowest point is ω, the tension at the ends is 3ω, where ω is the weight of unit length of the string.

11. A heavy string *just fits* round a vertical circle; show that the tension at the highest point is three times that at the lowest. ['just fits' means pressure at some point is zero].

12. A uniform string rests on a smooth cycloidal curve whose axis is vertical and vertex upwards, the string just reaching to the cusps of the cycloid. Prove that the pressure at any point varies inversely as the curvature.

13. A heavy uniform string hangs beneath a smooth cycloidal arc whose base is horizontal and vertex downwards; what must be the tension at the cusps in order that the chain may be everywhere in contact?

14. A heavy uniform string rests on a smooth catenary with its axis vertical and its vertex upwards; prove that the pressure on the curve at any point varies inversely as the radius of curvature at the point. Conversely if the curve be such that the pressure at every point varies inversely as the radius of curvature, then the curve is a catenary.

15. A uniform string rests in limiting equilibrium on a rough quadrant of a vertical circle with one end at the highest point. Prove that $\frac{\pi}{2}\tan\lambda = \log\tan(2\lambda)$, where λ is the angle of friction.

16. A uniform string rests in limiting equilibrium on a rough vertical circle and subtends an angle β at the centre. Show that if α is the angular distance of its upper end from the highest point of the circle then $\cos(\alpha+\beta-2\lambda) = e^{\beta\tan\lambda}\cos(\alpha-2\lambda)$, where λ is the angle of friction.

17. A heavy uniform string rests on the upper surface of a rough vertical circle of radius a, and partly hangs vertically. If one end be at the highest point of the circle, show that the greatest length that can hang freely is $\dfrac{2a\mu+(\mu^2-1)ae^{\frac{\mu\pi}{2}}}{1+\mu^2}$.

18. A uniform string of weight W rests on a rough cycloid whose axis is vertical and vertex upwards, the string resting from the vertex to a cusp. Show that the least horizontal force applied at the vertex that will cause the string to slip upwards is $\dfrac{W(3e^{\frac{\mu\pi}{2}}-\mu^2-1)}{(\mu^2+4)}$.

19. A heavy uniform string AB is placed on the upper side of a rough catenary with its directrix horizontal and vertex downwards. If A is at the vertex, find the least force F which acting at B will just move the string.

20. A string rests in limiting equilibrium on a rough semi-circle, being acted on by a constant attractive force towards one of its extremities. Show that the coefficient of friction is given by $3\mu = (1 - 2\mu^2)e^{\mu\pi}$.

21. Find the law of repulsive force from the pole when, under its action, a string assumes the form of (*a*) a lemniscate $r^2 = a^2 \cos 2\theta$; (*b*) a cardioid $r = a(1 + \cos \theta)$; (*c*) an equiangular spiral $r = ae^{\theta \cot \alpha}$; (*d*) a circle, the pole being (1) a point of the circumference (2) any point in the plane.

22. A string is in equilibrium under the action of a central force. If F be the force at any point per unit of length, prove that the tension at that point $= 2Fc$, where c is the chord of curvature through the centre of force. Show also that $F \propto \frac{r}{p^2\rho}$.

23. If a uniform string assume the form of an ellipse under the action of a repulsive force from the centre, find the law of force.

24. Show that an equiangular spiral is a possible form for a catenary of uniform strength, whose ends are fixed, to assume under a repulsive force from the pole varying inversely as the distance.

25. $ABCD$ is a square of side b. A uniform string of line-density δ, fixed at B and D, is in equilibrium under a repulsive force μr^{-3} from A. If the tangents to the string at B and D are perpendicular to BD and if the tension at each of these points is $\frac{\mu\delta}{2b^2}$, show that the shape of the string is the curve $r = b(\sin \theta + \cos \theta)$.

26. An infinite string passes through two small smooth rings, and is acted on by a force tending from a given fixed point and varying inversely as the cube of the distance from that point. Show that the part of the string between the rings assumes the form of an arc of a circle.

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CHAPTER VII

FORCES IN THREE DIMENSIONS

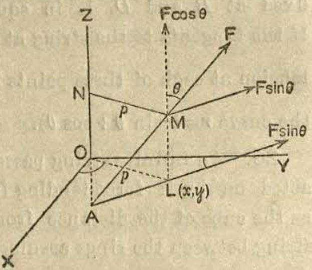
7·1. Moment of a force about a line.

Definition. Let F be any force and OZ any line in space. Let θ be the angle between them and resolve F into $F\cos\theta$ parallel to OZ and $F\sin\theta$ perpendicular to OZ (See fig. Art. 7·2). Let p be the shortest distance between F and OZ. Then $F\sin\theta.p$ is called the moment of F about OZ. Thus we have the definition : *The moment of a force about a line is the product of the resolved part of the force perpendicular to the line and the shortest distance between the force and the line.*

7·2. Theorem.

If two forces act at a point, the algebraic sum of their moments about any line in space is equal to the moment of their resultant about the same line.

Let F be any force and OZ any line in space. Through any point A on F draw the plane $XOY \perp$ to OZ cutting it at O. Let AL be the projection of AM, the line of action of F, on the plane XOY and draw $OL \perp$ on AL. Then OL is p, the s. d. between F and OZ. The projection of F is $F\sin\theta$, along AL where θ is the angle between F and OZ. Its moment about $O = F\sin\theta.OL = F\sin\theta.p =$ moment of F about OZ.



Thus the moment of F about OZ is equal to the moment of its projection on the plane XOY about O. Now, let F_1, F_2 be two forces acting at A and let R be their resultant. Let F'_1, F'_2, R' be their projections on the plane XOY, then R' is the resultant of F'_1 and F'_2, because a parallelogram projects into a parallelogram. Therefore, the algebraic sum of the moments of F_1 and F_2 about OZ

= the algebraic sum of moments of F'_1 and F'_2 about O

= moment of R' about O (by Varignon's theorem).

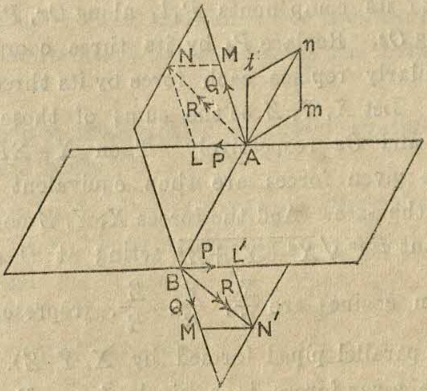
= moment of R about OZ. This proves the theorem.

7·3. Axis of a couple.

We know couples acting in the same plane and having equal moments are equivalent. So to give a couple in a plane it is not necessary to give the positions or magnitudes of the forces of the couple but only the moment of the couple should be given. Again, couples acting in parallel planes and having equal moments are equivalent ; so it is not even necessary to state the plane of the couple, it is enough to give the direction of its plane. Thus a couple is completely determined by the direction of its plane of action and its moment and its sense. The *Axis of a couple* is a directed straight line drawn from any point in space perpendicular to the plane of the couple, its length representing the moment of the couple and its direction, determined by the right hand screw rule, giving the sense of the couple. *A couple is thus completely represented by its axis.* Couples having equal axes are completely identical. *The axis of a couple is a free vector, and obeys the law of Vector Addition*, as will be seen from the following theorem.

7·4. To find the resultant of any two couples acting on a body.

Let the two couples act in the planes π and π' which intersect in AB. The two couples can be replaced in their own planes, by two other couples, each having the same arm AB, the forces of the first couple being P, P represented by AL and BL' and the



forces of the second couple being Q, Q represented by AM and BM', where $P.AB$ and $Q.AB$ are equal to the moments of the given couples. The two given couples are therefore equivalent to the four forces P, Q at A and P, Q at B. The resultant of P and Q at A is R represented by AN, the diagonal of the parallelogram $ALNM$; and the resultant of P and Q at B is an equal and unlike parallel force R represented by BN'. Therefore the resultant of the two couples is a single couple (R, R) of moment $R.AB$. Rotate the parallelogram $ALNM$ in its own plane about A through 90° so that it comes to the position $Alnm$. Then Al, Am and An are the axes of the three couples (P, P), (Q, Q) and (R, R). For they are respectively perpendicular to the planes and proportional to the moments $P.AB$, $Q.AB$ and $R.AB$ of these three couples and they have the directions determined by the right hand screw rule. The axis An of the resultant couple (R, R) is the diagonal of the parallelogram formed by Al and Am the axes of the given couples. Hence *the resultant of any two couples is a couple whose axis is found from the axes of the component couples by the parallelogram law.* It follows, therefore that *couples represented by their axes can be compounded and resolved in the same way as forces.* In other words, couples are free vectors and obey the vector law of addition.

7·5. Resultant of any number of forces acting at a point.

Let any number of forces P_1, P_2,...act at a point O. We take O as origin of any rect. axes Ox, Oy, Oz. Let (l_1, m_1, n_1), (l_2, m_2, n_2),...be the direction cosines of $P_1, P_2, \ldots$ Then P_1 is equivalent to its components P_1l_1 along Ox, P_1m_1 along Oy and P_1n_1 along Oz. Replace P_1 by its three components along the axes. Similarly replace each force by its three components along the axes. Let X, Y, Z be the sums of these components along Ox, Oy and Oz respectively. Then $X=\Sigma Pl$, $Y=\Sigma Pm$, $Z=\Sigma Pn$. The given forces are then equivalent to the forces X, Y, Z along the axes. And the forces X, Y, Z compound into a single resultant $R=\sqrt{X^2+Y^2+Z^2}$ acting at O along a line whose direction cosines are $\frac{X}{R}$, $\frac{Y}{R}$, $\frac{Z}{R}$. (represented by the diagonal of the parallelopiped formed by X, Y, Z). Hence the resultant of the given forces is a single force R at O whose magnitude and direction are as given above.

7·6. Resultant of any number of couples acting on a rigid body.

Couples are represented by their axes. We take any point O as origin of any rectangular axes Ox, Oy, Oz. Let the axes of the given couples be all drawn through O. Since axes can be resolved and compounded like forces, let all the axes be resolved along Ox, Oy, Oz. Let L, M, N be the sums of the resolved parts of the axes along Ox, Oy, Oz respectively. Then L, M, N can be compounded by the parallelopiped law, into the axis of the resultant couple G, whose moment is therefore, $\sqrt{L^2+M^2+N^2}$ and whose axis has direction cosines $\left(\frac{L}{G}, \frac{M}{G}, \frac{N}{G}\right)$. The resultant couple G is the resultant of the given couples. *Thus the resultant of any number of couples is a couple.*

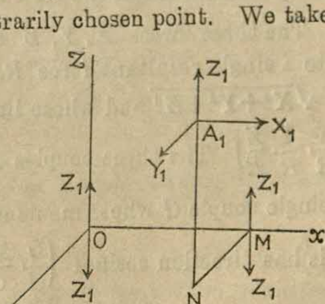
7·7. Reduction of a system of forces acting on a rigid body.

Theorem. *Anys ystem of forces acting on a rigid body can be reduced to a single force acting at an arbitrarily chosen point, together with a couple.*

Take any rectangular axes Ox, Oy, Oz where O is an arbitrarily chosen point. Each force of the system is replaced by an equal and parallel force acting at O and a couple. All the forces acting at O can be compounded into a single resultant force which acts at O and all the couples can be compounded into a single resultant couple. Thus the system reduces to a single force at O and a single couple. The resultant force is called the *Principal force* at O and the resultant couple is called the *Principal couple* at O.

7·8. To reduce analytically a given system of forces acting on a rigid body.

Let the forces be P_1, P_2, ... acting at the points A_1, A_2, ... of a rigid body. Let O be any arbitrarily chosen point. We take O as origin and any three mutually perpendicular lines Ox, Oy, Oz as rectangular axes. Let (x_1, y_1, z_1), (x_2, y_2, z_2),...be the co-ordinates of the points A_1, A_2, ... and let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), ... be the components of the forces P_1, P_2, ... parallel to the axes. Let $OM = x_1$, $MN = y_1$, $NA_1 = z_1$. At each of the points M and O apply two opposite forces each equal and parallel to Z_1. The introduced



forces form a null system and do not affect the resultant of the system.

The five forces, each equal to Z_1, are equivalent to a force Z_1 along Oz and two couples of arms NM and OM. The moment of the first couple is y_1Z_1 and the axis is Ox and the moment of the second couple is $-x_1Z_1$ and the axis is Oy (the system is left-handed) Thus the force Z_1 at A_1 is equivalent to a force Z_1 along Oz and two couples y_1Z_1 along Ox and $-x_1Z_1$ along Oy. In a similar manner, we find that the force X_1 at A_1 is equivalent to a force X_1 along Ox and two couples z_1X_1 along Oy and $-y_1X_1$ along Ox and the force Y_1 at A_1 is equivalent to a force Y_1 along Oy and two couples x_1Y_1 along Oz and $-z_1Y_1$ along Ox. Thus the force P_1 at A_1 can be replaced by the three components X_1, Y_1, Z_1 acting at O along Ox, Oy, Oz and the three couples of moments $(y_1Z_1-z_1Y_1)$, $(z_1X_1-x_1Z_1)$, $(x_1Y_1-y_1X_1)$ whose axes are Ox, Oy, Oz respectively. These moments are the moments of P_1 at A_1 about the axes of x, y and z respectively.

Let all the forces of the system be thus replaced Let $X=\Sigma X_1$, $Y=\Sigma Y_1$, $Z=\Sigma Z_1$, $L=\Sigma(y_1Z_1-z_1Y_1)$, $M=\Sigma(z_1X_1-x_1Z_1)$, $N=\Sigma(x_1Y_1-y_1X_1)$ Then the given system is equivalent to the three forces X, Y, Z acting at O along Ox, Oy, Oz and the three couples of moments L, M, N with axes along Ox, Oy, Oz, where $X=$sum of the resolved parts of the forces along Ox with similar meaning for Y and Z and $L=$sum of the moments of the forces about Ox, with similar meaning for M and N. These are called the *six components* of the given forces.

The three forces X, Y, Z acting at O can be compounded into a single resultant force R, acting at O, whose magnitude is $\sqrt{X^2+Y^2+Z^2}$ and whose line of action has direction cosines $\left(\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}\right)$. The three couples L, M, N can be compounded into a single couple G whose moment is $\sqrt{L^2+M^2+N^2}$ and whose axis has direction cosines $\left(\frac{L}{G}, \frac{M}{G}, \frac{N}{G}\right)$.

Thus the given system has finally been reduced to a single force R acting at an arbitrarily chosen point O and a couple G. This combination of a force and a couple has been called by

Plücker a *dyname* and the six quantities X, Y, Z, L, M, N, are the components of the dyname. The point O with respect to which the system is reduced is called the base of reference or simply the base.

7·9. R is an invariant of the system but not G.

Since R is the resultant of the given forces moved parallel to themselves to act at the base O, it follows that R is the same in magnitude and direction for all bases. Hence $X^2+Y^2+Z^2=R^2$ does not vary with the change of base or axes. It is called *an invariant* of the system.

But the couple G depends on the base chosen. For let O' be any other base. Then R at $O\equiv R$ at $O'+$a couple (R, R). The resultant of the two couples G and (R, R) will be a couple $G'\neq G$. Thus with respect to the base O', the system reduces to the force R at O' and the couple G' and $G'\neq G$. Hence G is not an invariant.

7·10. If instead of reducing the given system with respect to the origin O, we reduce it with respect to any other point O', whose co-ordinates are (ξ, η, ζ), then to find the values of the six components of the system, we transfer the origin to O'. This changes the co-ordinates of A_1, A_2......to $(x_1-\xi, y_1-\eta, z_1-\zeta)$, etc. but does not change the components $(X_1\ Y_1\ Z_1)$ etc. of the forces. Hence X, Y, Z are not altered, but L, M, N change to L', M', N', where

$$L'=\Sigma[y_1-\eta)Z_1-(z_1-\zeta)Y_1] =\Sigma(y_1Z_1-z_1Y_1)-\eta\Sigma Z_1+\zeta\Sigma Y_1$$

$$=L-\eta Z+\zeta Y.$$

$$M'=\Sigma[(z_1-\zeta)X_1-(x_1-\xi)Z_1]=M-\zeta X+\xi Z.$$

$$N'=\Sigma[x_1-\xi)Y_1-(y_1-\eta)X_1]=N-\xi Y+\eta X.$$

Thus L, M, N change and therefore G changes when the base changes, but R does not.

7·11. Theorem. *A force and a couple cannot produce equilibrium.*

For let the force be R acting at O and let G be the couple. Let OA be the axis of G and through O draw the plane π perpendicular to OA. Then G acts in the plane π and one of its forces P may be taken to act at O. The resultant of R and P acting at O is a single force which acts at O and cannot be balanced by the other force P of the couple G which does not pass through O.

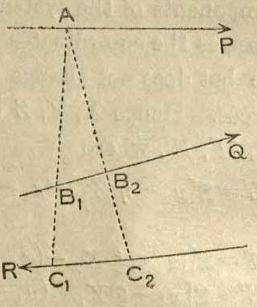
7·12. Conditions of equilibrium of a system of forces acting on a body.

Let the system be reduced to a force R and a couple G with respect to any base O and any rect axes Ox, Oy, Oz. Since R and G cannot produce equilibrium, it is necessary and sufficient for equilibrium that we should have $R=0$ and $G=0$.

As $R^2=X^2+Y^2+Z^2$ and $G^2=L^2+M^2+N^2$, for R and G to be zeros we must have $X=0$, $Y=0$, $Z=0$ and $L=0$, $M=0$, $N=0$. Since the origin and the axes are arbitrary it follows that for equilibrium it is necessary and sufficient that (1) *the sums of the resolved parts of the forces along three mutually perpendicular lines should separately vanish* and (2) *the sums of the moments of the forces about three mutually perpendicular lines should separately vanish.* So *the forces have no component along any line and no moment about any line.*

7·13. *If three forces acting on a body keep it in equilibrium, they must be coplanar.*

Let P, Q, R be the three forces which act on a body and are in equilibrium. Take *any* point A on P and two points B_1, B_2 on Q. Since the sum of the moments of P, Q, R about the line AB_1 is zero and the moments of P and Q about AB_1 are individually zero as they meet AB_1, it follows that the moment of R about AB_1 is zero. $\therefore$ AB_1 cuts R at C_1, say. Similarly AB_2 cuts R at C_2, say. $\therefore$ Q and R are coplanar lying in plane (B_1C_1, B_2C_2) and in that plane lies A. As A is any point on P and the plane of Q and R is fixed, it follows that P lies in the plane of Q and R. Hence the forces P, Q, R are coplanar.

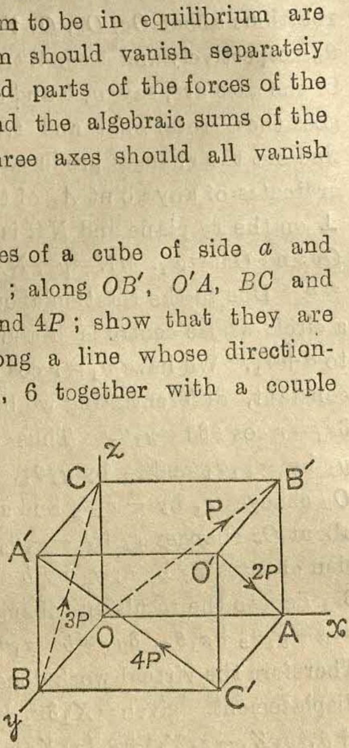


7·14. Deduction of the conditions of equilibrium of a system of forces acting on a rigid body from the Principle of Virtual Work.

Let P_1, P_2,...be the forces of the system acting at the points A_1, A_2,...of the body. Let a point O of the body be chosen as origin and three mutually perpendicular lines Ox, Oy, Oz as axes of co-ordinates, which are fixed in space. Let (x_1, y_1, z_1), (x_2, y_2, z_2),$\cdots$be the co-ordinates of the points A_1, A_2.$\cdots$and let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2),$\cdots$be the components of the forces P_1, P_2,$\cdots$parallel to the axes. Let the body be given any arbitrary small virtual displacement consistent with the geometrical condition of the system. We know that the general displacement of a rigid body consists of a rotation about an axis passing through any of its points and a translation, which rotation and translation are the same for all its points. Again, we know that any rotation about an axis may be resolved into three component rotations about three axes, not in same plane, by the Parallelogram law. Hence the arbitrary small displacement given to the body consists of three component rotations about Ox, Oy, Oz through small angles, say, θ_1, θ_2 and θ_3 respectively and a small translation whose components parallel to the axes are, say, a, b and c. Every point of the body undergoes these rotational and translational displacements. Let us find out the changes produced by these displacements in the co-ordinates of any point A_1 of the body (see fig. Art 7·8). A_1N is $\perp$ on the xy plane and NM is $\perp$ on Ox. Therefore A_1M is $\perp$ on Ox and $OM=x_1$, $MN=y_1$, $NA_1=z_1$. Let $A_1M=r$ and $\angle A_1MN=\theta$. Due to the rotation about Ox, A_1 describes a small arc of a circle whose centre is M and radius is $A_1M=r$ and θ changes to $\theta+\theta_1$. We have $y_1=r\cos\theta$ and $z_1=r\sin\theta$. Since r remains constant, differentiating we have, $\delta y_1=-r\sin\theta\,\delta\theta=-z_1\theta_1$ and $\delta z_1=r\cos\theta\delta\theta=y_1\theta_1$. Thus a rotation of θ_1 about Ox changes y_1 by $-z_1\theta_1$ and z_1 by $y_1\theta_1$. Similarly a rotation of θ_2 about Oy changes z_1 by $-x_1\theta_2$ and x_1 by $z_1\theta_2$ and a rotation of θ_3 about Oz changes x_1 by $-y_1\theta_3$ and y_1 by $x_1\theta_3$; and the translation changes x_1 by a, y_1 by b and z_1 by c. Hence if δx_1, δy_1, δz_1 denote the resultant changes in x_1, y_1, z_1, we have $\delta x_1=a-y_1\theta_3+z_1\theta_2$, $\delta y_1=b-z_1\theta_1+x_1\theta_3$ and $\delta z_1=c-x_1\theta_2+y_1\theta_1$. Therefore the virtual work done by the force P_1 in the virtual displacement given$=X_1\delta x_1+Y_1\delta y_1+Z_1\delta z_1=aX_1+bY_1+cZ_1+\theta_1(y_1Z_1-z_1Y_1)+\theta_2(z_1X_1-x_1Z_1)+\theta_3(x_1Y_1-y_1X_1)$. Since $a, b, c, \theta_1, \theta_2, \theta_3$ are the same for all the points A_1, A_2, ..., the virtual work done by the system in the virtual displacement given$=a\Sigma X_1+b\Sigma Y_1+c\Sigma Z_1+\theta_1\Sigma(y_1Z_1-z_1Y_1)+\theta_2\Sigma(z_1X_1-$

$x_1Z_1)+\theta_3\Sigma(x_1Y_1-y_1X_1)=aX+bY+cZ+\theta_1L+\theta_2M+\theta_3N\cdots(1)$, where X, Y, Z, L, M, N are the six components of the system with respect to base O. First suppose the system is in equilibrium; then by the principle of virtual work, the work done by the system $aX+bY+cZ+\theta_1L+\theta_2M+\theta_3N$ is zero for all arbitrary small values of $a, b, c, \theta_1, \theta_2, \theta_3$. Therefore $X=o$, $Y=o$, $Z=o$, $L=o$, $M=o$, $N=o$. Secondly suppose X, Y, Z, L, M, N are all zeros. Then from (1), the virtual work done by the system is zero whatever values $a, b, c, \theta_1, \theta_2, \theta_3$ may have provided they are small. This means that the virtual work of the system is zero for all arbitrary small virtual displacements, consistent with the geometrical condition of the system; and therefore by the converse of the principle of virtual work it follows that the system is in equilibrium. Hence the necessary and sufficient conditions for the system to be in equilibrium are that the six components of the system should vanish separately i.e. the algebraic sums of the resolved parts of the forces of the system parallel to the three axes and the algebraic sums of the moments of the forces about the three axes should all vanish separately.

Example. OA, OB, OC are edges of a cube of side a and OO', AA', BB', CC' are its diagonals; along OB', $O'A$, BC and $C'A'$ act forces equal to P, $2P$, $3P$ and $4P$; show that they are equivalent to a force $\sqrt{35}P$ at O along a line whose direction-cosines are proportional to $-3, -5, 6$ together with a couple $\frac{Pa}{2}\sqrt{114}$ about a line whose direction-cosines are proportional to $7, -2, 2$.



If in a system of forces, a force acts at (x_1, y_1, z_1) and has components (X_1, Y_1, Z_1), ...then they reduce to a force R at O whose components are (X, Y, Z) where $X=\Sigma X_1$, $Y=\Sigma Y_1$, $Z=\Sigma Z_1$ and to a couple G whose components are (L, M, N) where $L=\Sigma(y_1Z_1-z_1Y_1)$, $M=\Sigma(z_1X_1-x_1Z_1)$,

$$N=\Sigma(x_1Y_1-y_1X_1).$$

Here, the co-ordinates of the corners of the cube are $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $A'(0, a, a)$, $B'(a, 0, a)$, $C'(a, a, 0)$ and $O'(a, a, a)$.

$\therefore$ direction-cosines of $\overrightarrow{OB'}$ are $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, of $\overrightarrow{O'A}$ are $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, of $\overrightarrow{BC}$ are $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and of $\overrightarrow{C'A'}$ are $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. Now, we write the co-ordinates of the point of application of each force and its components in two rows, one just below the other, thus :

$$(\text{ for } P \text{ along } OB'\ )\begin{pmatrix} 0 & 0 & 0 \\ \frac{P}{\sqrt{2}} & 0 & \frac{P}{\sqrt{2}} \end{pmatrix}$$

$$(\text{ for } 2P \text{ along } O'A\ )\begin{pmatrix} a & 0 & 0 \\ 0 & \frac{-2P}{\sqrt{2}} & \frac{-2P}{\sqrt{2}} \end{pmatrix}$$

$$(\text{ for } 3P \text{ along } BC\ )\begin{pmatrix} 0 & a & 0 \\ 0 & \frac{-3P}{\sqrt{2}} & \frac{+3P}{\sqrt{2}} \end{pmatrix}$$

$$(\text{ for } 4P \text{ along } C'A'\ )\begin{pmatrix} a & a & 0 \\ \frac{-4P}{\sqrt{2}} & 0 & \frac{4P}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \quad X=\frac{-3P}{\sqrt{2}},\ Y=\frac{-5P}{\sqrt{2}},\ Z=\frac{6P}{\sqrt{2}}$$

$\therefore \quad R=\sqrt{X^2+Y^2+Z^2}=\frac{P}{\sqrt{2}}\sqrt{10}=P\sqrt{35}$ and d-cosines are proportional to X, Y, Z *i.e.* $-3, -5, 6$.

And $L=\frac{7aP}{\sqrt{2}}$, $M=\frac{-2aP}{\sqrt{2}}$, $N=\frac{2aP}{\sqrt{2}}$

$\therefore \quad G=\sqrt{L^2+M^2+N^2}=\frac{aP}{2}\sqrt{114}$ and d-cosines of its axis are proportional to L, M, N *i.e.* $7, -2, 2$.

Hence the system reduces to a force $\sqrt{35}P$ whose direction numbers are $(-3, -5, 6)$ and to a couple of moment $\frac{aP}{2}\sqrt{114}$ whose axis has direction numbers $(7, -2, 2)$.

EXAMPLES

1. OA, OB, OC are the edges of a cube of side 2 ft. and OO', AA', BB', CC' are its diagonals ; along OB', $O'A$, BC and $C'A'$ act forces equal to 2, 6, 4 and 3 ℔. wt. Prove that they are equivalent to a force $\sqrt{55}$ ℔s. wt. at O in a direction whose direction cosines are proportional to -1, -10, 3 together with a couple $\sqrt{134}$ ft. ℔s. whose axis has direction cosines proportional to 7, 3, -3.

2. Six forces, each equal to P, act along the edges of a cube, taken in order, which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}.Pa$, where a is the edge of the cube.

3. Forces act through the angular points of a tetrahedron perpendicular to the opposite faces and proportional to them. Show that they are in equilibrium if they act all inwards or all outwards.

4. Forces perpendicular and proportional to the areas of the faces of a tetrahedron act at their respective centroids. Show that they are in equilibrium if they act all inwards or all outwards.

5. Four forces act along generators of the same system of a hyperboloid. Their magnitudes are such that if they were transferred parallel to themselves to act at one point they would be in equilibrium. Show that they are in equilibrium when acting along the generators.

6. Prove that the couples which act in and are proportional to the areas of the faces of a tetrahedron are in equilibrium if their axes are all outwards.

7. A regular tetrahedron formed of six light rods, each of length a, rests on a smooth horizontal plane. A ring, of weight W and radius b, is supported by the slant sides. Show that the stress in any one of the horizontal sides is $\frac{W}{\sqrt{6}}\left[\frac{1}{3}-\sqrt{3}.\,\frac{b}{a}\right]$.

8. A regular octahedron formed of twelve equal rods, each of weight ω, freely jointed together, is suspended from one corner. Show that the thrust in each horizontal rod is $\frac{3}{2}\omega\sqrt{2}$.

9. A conical tent resting on a smooth floor is made up of an indefinitely great number of equal isosceles triangular elements, hinged at the vertex and kept in shape by a heavy circular ring placed on it like a necklace. Show that in equilibrium the semi-vertical angle of the cone is $\sin^{-1}\left(\frac{r}{h}\cdot\frac{3W'}{W+3W'}\right)^{\frac{1}{3}}$, where W, W' are respectively the weights of the cone and the ring, and r, h are respectively the radius of the ring and the slant side of the cone.

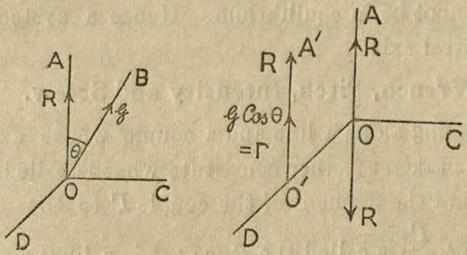
10. A tripod consists of three equal uniform bars, each of length a and weight ω, which are freely jointed at one extremity, their middle points being jointed by strings, each of length b. The tripod is placed with its free ends in contact with a smooth horizontal plane and a weight W is attached to the common joint; show that the tension of each string is

$$\frac{2}{3}(2W+3\omega)\frac{b}{\sqrt{9a^2-12b^2}}.$$

7·14. Poinsot's Central Axis

Theorem. *Any system of forces acting on a rigid body can be reduced to a single force and a couple whose axis lies along the line of action of the force.*

Take any point O and reduce the system w. r. to the base O to a force R acting at O and a couple G. Let R act along OA and let OB be the axis of G and let $\angle AOB=\theta$. In the plane AOB, draw $OC \perp$ to OA and draw $OD \perp$ to the plane AOB.



Then OC is $\perp$ to the plane AOD. Resolve G into $G\cos\theta$ along OA and $G\sin\theta$ along OC. The couple $G\sin\theta$ acts in the plane AOD in which the force R lies. It can be replaced by two equal unlike parallel forces R, R, at a distance $\dfrac{G\sin\theta}{R}$ apart. Let one of the forces R act at O in the direction AO. Then the other force R, ∥ to OA, will act along $O'A'$ at a point O' on OD such that $OO'=\dfrac{G\sin\theta}{R}$. The two forces R, R acting at O in opposite directions balance each other and we are left with the couple $G\cos\theta$ along OA and the force R at O' acting along $O'A'$ ∥ to OA. The axis of the couple $G\cos\theta$ can be transferred from OA to $O'A'$. Thus the system reduces to a force R at O' and a couple $G\cos\theta=\Gamma$, the axis of the couple coinciding with the line of action of the force R.

The axis $O'A'$ is called **Poinsot's Central Axis.**

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7·15. Theorem. *A given system of forces can have only one central axis.*

Whatever be the base w. r. to which the system is reduced the force component R is always the same in magnitude and direction. Now if possible let the system have two central axes i.e the system can be reduced to a force R acting at O along OA and a couple G whose axis is along OA and also to a force R acting at O' along $O'A'$ and a couple G' whose axis is along $O'A'$, where OA, $O'A'$ are parallel. Then the force R at O and the couple G will be in equilibrium with the reversed force R at O' and the reversed couple G'. But the two couples G and reversed G' are equivalent to a single couple $G-G'$ with axis along OA and the force R at O with the reversed force R at O' form also a couple whose axis is perpendicular to OA, the axis of the first couple. The two couples, having their axes in perpendicular directions cannot be in equilibrium. Hence a system can have only one central axis.

7·16. Wrench, Pitch, Intensity and Screw.

A force acting along a line and a couple whose axis coincides with this line taken together constitute what is called a *wrench*.

The ratio of the moment of the couple Γ to the magnitude of the force R *i.e.* $\frac{\Gamma}{R}$ is called the *Pitch* ; it is a linear magnitude.

The force which acts in a wrench is called the *intensity* of the wrench.

The straight line along which the force acts in a wrench considered together with the pitch is called a *Screw*. Five quantities are required to determine a screw, four to determine the straight line and the fifth to determine the pitch.

The straight line along which the force acts in a wrench is called the *axis of the wrench.*

A force alone may be regarded as a wrench of zero pitch and a couple alone may be regarded as a wrench of infinite pitch.

7·17. To find the condition that a given system of forces may have a single resultant.

Let the system be reduced to a single force R acting at an arbitrary base O and a couple G. If the force R lies in the plane of the couple G, then we know the force R and the couple G are together equivalent to a single force, equal and parallel to R. But if the force R does not lie in the plane of the couple G, then the force R intersects the plane of the couple at O and we can replace the couple G by two equal and opposite forces P, P in its plane, only one of which passes through O; then the two forces R and P at O will combine into a single force at O but not lying in the plane. This single force and the other force P do not meet nor are they parallel, so they cannot reduce to a single force. Thus the system can reduce to a single force if and only if the force R lies in the plane of the couple G and then the line of action of R and the axis of the couple G are perpendicular. The direction cosines of R are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and those of G are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$.

$\therefore$ The condition is $LX+MY+NZ=0$, provided X, Y, Z do not all vanish. For in that case $R=0$ and the system reduces to the couple G. But if $G=0$, then the system reduces to a single force R and $LX+MY+NZ=0$.

7·18. Theorem.

Invariants of a given system of forces—*Whatever may be the base or the axes chosen to reduce the system, the quantities* $X^2+Y^2+Z^2$ *and* $LX+MY+NZ$ *are invariable i.e. they always remain the same.*

We take any point O as origin and choose any rect. axes. Let the components of the system be X, Y, Z, L, M, N referred to O as base. Let R be the resultant force and G the resultant couple. We know the given system of forces can be reduced to a wrench in which the force is R and the couple is $G \cos \theta = \Gamma$, where θ is the angle between the line of action of R and the axis of the couple G; and we have seen that the reduction to a wrench is unique, *i.e.* by whatever process we reduce the given system to a wrench the force is always R and the couple is always Γ. We know otherwise that R must be invariable for it is the resultant of the given forces transferred parallel to themselves to the bass O and now we find that Γ must also be invariable. $\therefore$ R and Γ are invariants of the system. We have $R^2=X^2+Y^2+Z^2$ and

$$\Gamma = G \cos \theta = G\cdot\left(\frac{X}{R}\cdot\frac{L}{G}+\frac{Y}{R}\cdot\frac{M}{G}+\frac{Z}{R}\cdot\frac{N}{G}\right)=\frac{LX+MY+NZ}{R}.$$

[$\because$ direction cosines of R are $\left(\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}\right)$ and those of the axis of G are $\left(\frac{L}{G}, \frac{M}{G}, \frac{N}{G}\right)$.]

$\therefore$ $X^2+Y^2+Z^2$ and $LX+MY+NZ$ are invariants of the system.

7·19. To find the equation of the central axis of a given system of forces.

With reference to any base O and rectangular axes Ox, Oy, Oz let the system be reduced and let the six components of the system be X, Y, Z, L, M, N. Let O' (ξ, η, ζ) be any point on the central axis, then if the system be reduced w. r. to O' as base, the resultant force will be R and the couple will be Γ and the force R and the axis of the couple Γ will have the same direction. Now, w. r. to O' as base, the six components of the system are X, Y, Z, L', M', N' where

$$L'=\Sigma[(y_1-\eta)Z_1+(z_1-\zeta)Y_1]=L-\eta Z+\zeta Y$$
$$M'=\Sigma[(z_1-\zeta)X_1-(x_1-\xi)Z_1]=M-\zeta X+\xi Z$$
$$N'=\Sigma[(x_1-\xi)Y_1-(y_1-\eta)X_1]=N-\xi Y+\eta X.$$

[See Art. 7·10]

The direction cosines of R are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$

and those of the axis of Γ are $\frac{L'}{\Gamma}, \frac{M'}{\Gamma}, \frac{N'}{\Gamma}$.

Since R and the axis of Γ are parallel, we have,

$$\frac{L'}{X}=\frac{M'}{Y}=\frac{N'}{Z}.$$

$$or, \quad \frac{L-\eta Z+\zeta Y}{X}=\frac{M-\zeta X+\xi Z}{Y}=\frac{N-\xi Y+\eta X}{Z}$$

$\therefore$ Equations of the central axis, which is the locus of (ξ, η, ζ), are $\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}$

$$=\frac{LX+MY+NZ}{R^2}$$

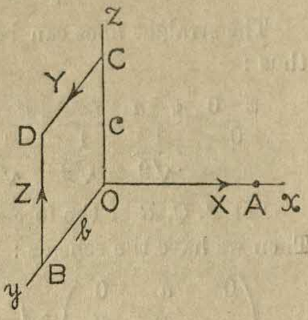
$$=\frac{\Gamma}{R}.$$

Examples Worked

Ex. 1. $OBDC$ is a rectangle such that $OB=b$ and $OC=c$; also OA is perpendicular to its plane; along OA, CD and BD act forces X, Y and Z respectively. Show that the component force R and couple k of the resultant wrench are $\sqrt{X^2+Y^2+Z^2}$ and $\dfrac{X(Zb-Yc)}{\sqrt{X^2+Y^2+Z^2}}$. Show also that with OA, OB and OC as axes of x, y and z the equation to the central axis is

$$\frac{x}{X}=\frac{y}{Y}-\frac{Zk}{XYR}=\frac{z}{Z}+\frac{kY}{XZR}.$$

We take OA, OB, OC as axes of x, y and z. The co-ordinates of O, B and C are $(0, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. We write the co-ordinates of the point of application of each given force and its components in two rows one just below the other thus;



$$\begin{pmatrix}0&0&0\\X&0&0\end{pmatrix};\ \begin{pmatrix}0&b&0\\0&0&Z\end{pmatrix};\ \begin{pmatrix}0&0&c\\0&Y&0\end{pmatrix}.$$

Hence $\Sigma X=X,\ \Sigma Y=Y,\ \Sigma Z=Z \quad \therefore\ R=\sqrt{X^2+Y^2+Z^2}$

And $L=bZ-cY,\ M=0,\ N=0 \quad \therefore\ G=bZ-cY.$

Now we know, $kR=LX+MY+NZ$

$$\therefore\quad k=\frac{(bZ-cY)X}{\sqrt{X^2+Y^2+Z^2}}.$$

Equation of the central axis is

$$\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}=\frac{k}{R}.$$

$$or\quad \frac{(bZ-cY)-yZ+zY}{X}=\frac{-zX+xZ}{Y}=\frac{-xY+yX}{Z}=\frac{k}{R}.$$

From last two, $xYR=yXR-kZ$; dividing by XYR, we have,

$$\frac{x}{X}=\frac{y}{Y}-\frac{kZ}{XYR}$$

From 2nd & 4th, $xZR=zXR+kY$; dividing by XZR, we have

$$\frac{x}{X}=\frac{z}{Z}+\frac{kY}{XZR}$$

Hence the equation of central axis is

$$\frac{x}{X}=\frac{y}{Y}-\frac{kZ}{XYR}=\frac{z}{Z}+\frac{kY}{XZR}.$$

Ex. 2. Three forces act along the straight lines $x=0$, $y-z=a$; $y=0$, $z-x=a$; $z=0$, $x-y=a$. Show that they cannot reduce to a couple. Prove also that if the system reduces to a single force its line of action must lie on the the surface

$$x^2+y^2+z^2-2yz-2zx-2xy=a^2. \qquad \text{[C. H. 1965]}$$

The straight lines can be written in the symmetrical form thus :

$$\frac{x-0}{0}=\frac{y-a}{\frac{1}{\sqrt{2}}}=\frac{z-0}{\frac{1}{\sqrt{2}}}\ ;\ \frac{x-0}{\frac{1}{\sqrt{2}}}=\frac{y-0}{0}=\frac{z-a}{\frac{1}{\sqrt{2}}}\ ;\ \frac{x-a}{\frac{1}{\sqrt{2}}}=\frac{y-0}{\frac{1}{\sqrt{2}}}=\frac{z-0}{0}$$

Let P, Q, R be the forces acting along the lines respectively. Then we have the scheme :

$$\begin{pmatrix} 0 & a & 0 \\ 0 & \frac{P}{\sqrt{2}} & \frac{P}{\sqrt{2}} \end{pmatrix};\ \begin{pmatrix} 0 & 0 & a \\ \frac{Q}{\sqrt{2}} & 0 & \frac{Q}{\sqrt{2}} \end{pmatrix};\ \begin{pmatrix} a & 0 & 0 \\ \frac{R}{\sqrt{2}} & \frac{R}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\therefore \quad X=\frac{Q+R}{\sqrt{2}},\ Y=\frac{R+P}{\sqrt{2}},\ Z=\frac{P+Q}{\sqrt{2}}$$

$$\therefore \quad R^2=\tfrac{1}{2}[(Q+R)^2+(R+P)^2+(P+Q)^2].$$

If R be zero, we must have $Q+R=0$, $R+P=0$, $P+Q=0$

$$\therefore \quad P=Q=R=0.$$

So R cannot be zero and hence the system cannot reduce to a couple.

If the system reduces to a single force then $LX+MY+NZ=0$ and the line of action of the resultant is the central axis.

Now, $L=\frac{aP}{\sqrt{2}}$, $M=\frac{aQ}{\sqrt{2}}$, $N=\frac{aR}{\sqrt{2}}$

$\therefore$ Equations of the central axis are

$$\frac{\frac{aP}{\sqrt{2}}-y\left(\frac{P+Q}{\sqrt{2}}\right)+z\left(\frac{R+P}{\sqrt{2}}\right)}{X}=\cdots\cdots\cdots\cdots=0$$

$$\begin{aligned} \text{or} \quad aP-y(P+Q)+z(R+P)&=0 \\ aQ-z(Q+R)+x(P+Q)&=0 \\ aR-x(R+P)+y(Q+R)&=0 \end{aligned}$$

$$\text{or}\quad (a-y+z)P-yQ+zR=0$$
$$xP+(a-z+x)Q-zR=0$$
$$-xP+yQ+(a-x+y)R=0$$

Eliminating P, Q and R we get the locus of the central axis in the determinant form :

$$\begin{vmatrix} a-y+z & -y & z \\ x & a-z+x & -z \\ -x & y & a-x+y \end{vmatrix}=0$$

$$\text{or}\quad \begin{vmatrix} a & -y & z \\ 2z-a & a-z+x & -z \\ -2y-a & y & a-x+y \end{vmatrix}=0$$

(Subtracting from col 1 the sum of cols 2 and 3).

$$\text{or}\quad \begin{vmatrix} a & -y & z \\ 2z & a-z+x-y & 0 \\ -2y & 0 & a-x+y+z \end{vmatrix}=0 \quad \text{(adding row 1 to the 2nd and 3rd rows)}$$

or expanding,

$$a[a^2-(x-y-z)^2]+2yz(a-x+y+z)+2yz(a-z+x-y)=0$$

$$\text{or}\quad a[a^2-(x-y-z)^2+4yz]=0$$

$$\text{or}\quad (x-y-z)^2-4yz=a^2$$

$$\text{or}\quad x^2+y^2+z^2-2yz-2zx-2xy=a^2.$$

Ex. 3. A force P acts along the axis of x and another force nP acts along a generator of the cylinder $x^2+y^2=a^2$; show that the central axis lies on the cylinder $n^2(nx-z)^2+(1+n^2)^2y^2=n^4a^2$.

Suppose the force nP acts along the generator which is parallel to the z-axis, and which meets the xy-plane on the circle $x^2+y^2=a^2$, $z=0$ at the point $(a\cos\theta, a\sin\theta, 0)$

$\therefore$ Eqn. of the generator is $\dfrac{x-a\cos\theta}{0}=\dfrac{y-a\sin\theta}{0}=\dfrac{z-0}{1}$.

Now force P acts at $(0, 0, 0)$ and has components $(P, 0, 0)$. And force nP acts at $(a\cos\theta, a\sin\theta, 0)$ and has components $(0, 0, nP)$.

$\therefore$ We have the scheme : $\begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} a\cos\theta & a\sin\theta & 0 \\ 0 & 0 & nP \end{pmatrix}$.

$\therefore$ $X=P$, $Y=0$, $Z=nP$;
$L=a\sin\theta.nP$, $M=-a\cos\theta.nP$, $N=0$.

Equations of central axis are

$$\frac{an\sin\theta.P-ynP}{P}=\frac{-an\cos\theta.P-zP+xnP}{0}=\frac{yP}{nP}.$$

From 1st and last, $n^2.a\sin\theta=(n^2+1)y$ and from 2nd one, $an\cos\theta=nx-z$. Eliminating θ, the locus of the central axis is obtained as, $(n^2+1)^2y^2+n^2(nx-z)^2=(an^2)^2=a^2n^4$.

Ex. 4. Two forces act, one along the line $y=0$, $z=0$ and the the other along the line $x=0$, $z=c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is $(x^2+y^2)z=cy^2$. [C.H. 1962]

Let P, Q be the two forces acting along the first and the second given lines respectively. We have to determine, for each force, a point on its line of action and the direction-cosines of the latter. For this we write the equation of each line in the symmetrical form : $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0}$ and $\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-c}{0}$.

∴ P may be taken to act at $(0, 0, 0)$ and has components $(P, 0, 0)$ and Q may be taken to act as $(0. 0, c)$ and has components $(0, Q, 0)$

∴ we have the scheme : $\begin{pmatrix}0 & 0 & 0\\ P & 0 & 0\end{pmatrix}$; $\begin{pmatrix}0 & 0 & c\\ 0 & Q & 0\end{pmatrix}$;

∴ $X=P$, $Y=Q$, $Z=0$; $L=-cQ$, $M=0$, $N=0$.

∴ Equations of the central axis are,

$$\frac{-cQ+zQ}{P}=\frac{-zP}{Q}=\frac{-xQ+yP}{0}$$

or, $(P^2+Q^2)z=cQ^2$ and $\frac{x}{P}=\frac{y}{Q}$.

Eliminating $\frac{P}{Q}$ we get the locus of the central axis as

$$\left(\frac{x^2}{y^2}+1\right)z=c \quad or,\quad (x^2+y^2)z=cy^2.$$

Ex. 5. If a force (X, Y, Z) act along a generator of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\frac{z}{c}$, and be equivalent to an equal force (X, Y, Z) at the origin together with a couple (L, M, N), show that $aL\pm bM=0$, $bX\pm aY=0$ and $cN\pm abZ=0$.

Equation of a generator of the paraboloid is

$$\frac{x-a\lambda}{a}=\frac{y-b\lambda}{-b}=\frac{z-0}{2c\lambda} \quad \cdots \quad (1)$$

d. numbers of the line of action of given force are X, Y, Z.

$$\therefore \quad \frac{X}{a}=\frac{Y}{-b}=\frac{Z}{2c\lambda} \quad \cdots \quad (1) \qquad \text{Again } \begin{pmatrix} a\lambda & b\lambda & 0 \\ X & Y & Z \end{pmatrix}$$

$$\therefore \quad L=b\lambda Z,\ M=-a\lambda Z,\ N=(aY-bX)\lambda \quad \cdots \quad (2)$$

From (2), $\dfrac{L}{b}=\dfrac{M}{-a}$ and $N=a\left(-\dfrac{bZ}{2c}\right)-\dfrac{baZ}{2c}$ [from (1) & (2)]

or, $cN+abZ=0$.

Hence $bX+aY=0$, $aL+bM=0$ and $cN+abZ=0$.

Another generator is obtained by changing sign of b in (1). If given force acts along this other generator, we shall have, similarly, $-bX+aY=0$, $aL-bM=0$, $cN-abZ=0$.

Ex. 6. A force parallel to the axis of z acts at the point $(a, 0, 0)$ and an equal force perpendicular to the axis of z acts at the point $(-a, 0, 0)$. Show that the central axis of the system lies on the surface $z^2(x^2+y^2)=(x^2+y^2-ax)^2$.

Let P be the equal forces and let the second force, which lies in the xy-plane, make an angle θ with the x-axis so that its direction cosines are $(\cos\theta, \sin\theta, 0)$.

$$\therefore \quad \text{we have the scheme : } \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & P \end{pmatrix};\ \begin{pmatrix} -a & 0 & 0 \\ P\cos\theta & P\sin\theta & 0 \end{pmatrix}.$$

$$\therefore \quad X=P\cos\theta,\ Y=P\sin\theta,\ Z=P;$$
$$L=0,\ M=-aP,\ N=-aP\sin\theta.$$

$\therefore$ Equations of the central axis are,

$$\frac{-yP+zP\sin\theta}{P\cos\theta}=\frac{-aP-zP\cos\theta+xP}{P\sin\theta}$$
$$=\frac{-aP\sin\theta-xP\sin\theta+yP\cos\theta}{P}$$

$$\text{or,} \quad \frac{-y+z\sin\theta}{\cos\theta}=\frac{-a-z\cos\theta+x}{\sin\theta}$$
$$=\frac{-a\sin\theta-x\sin\theta+y\cos\theta}{1}=\frac{2(y\cos\theta-x\sin\theta)}{0}$$

(Multiplying the numerator and denominator of first by $\cos\theta$, of second by $\sin\theta$ and subtracting their sum from those of the last member).

From first two, $(x-a)\cos\theta+y\sin\theta=z$ ......(1)

From last one, $x\sin\theta=y\cos\theta$. $\therefore$ $\dfrac{\sin\theta}{y}=\dfrac{\cos\theta}{x}=k$, say

$\therefore\ \sin\theta=ky,\ \cos\theta=kx,$

squaring and adding, $1=k^2(x^2+y^2)$......(2)

From (1), $(x-a)kx+y.ky=z$ *or*, $(x^2+y^2-ax)k=z$......(3)

Eliminating k from (2) and (3), the locus of the central axis is $(x^2+y^2-ax)^2=(x^2+y^2)z^2$.

Ex. 7. Forces X, Y, Z act along the three straight lines $y=b, z=-c$; $z=c, x=-a$; $x=a, y=-b$, respectively ; show that they will have a single resultant if $\dfrac{a}{X}+\dfrac{b}{Y}+\dfrac{c}{Z}=0$;

[C. H. 1966]

and that the equations of its line of action are any two of the three $\dfrac{y}{Y}-\dfrac{z}{Z}-\dfrac{a}{X}=0,\ \dfrac{z}{Z}-\dfrac{x}{X}-\dfrac{b}{Y}=0,\ \dfrac{x}{X}-\dfrac{y}{Y}-\dfrac{c}{Z}=0.$

The equations of the lines written in symmetrical form are :

$$\frac{x-0}{1}=\frac{y-b}{0}=\frac{z+c}{0};\quad \frac{x+a}{0}=\frac{y-0}{1}=\frac{z-c}{0};\quad \frac{x-a}{0}=\frac{y+b}{0}=\frac{z-0}{1}.$$

$\therefore$ we have the scheme : $\begin{pmatrix}0 & b & -c\\ X & 0 & 0\end{pmatrix};\ \begin{pmatrix}-a & 0 & c\\ 0 & Y & 0\end{pmatrix};\ \begin{pmatrix}a & -b & 0\\ 0 & 0 & Z\end{pmatrix}$

$\therefore\ \Sigma X=X, \Sigma Y=Y, \Sigma Z=Z,$

$L=-cY-bZ, M=-cX-aZ, N=-bX-aY.$

The system will reduce to a single resultant if

$$LX+MY+NZ=0$$

or, $X(cY+bZ)+Y(aZ+cX)+Z(bX+aY)=0$

or, $aYZ+bZX+cXY=0$ *or*, $\dfrac{a}{X}+\dfrac{b}{Y}+\dfrac{c}{Z}=0$......(1)

Assuming this condition is satisfied, the equations of the line of action of the resultant are,

$$\frac{(-cY-bZ)-yZ+zY}{X}=\frac{(-aZ-cX)-zX+xZ}{Y}$$
$$=\frac{(-bX-aY)-xY+yX}{Z}=0$$

$$\left.\begin{aligned} \text{or,}\quad -\frac{c}{Z}-\frac{b}{Y}-\frac{y}{Y}+\frac{z}{Z}&=0,\\ -\frac{a}{X}-\frac{c}{Z}-\frac{z}{Z}+\frac{x}{X}&=0,\\ \text{and}\quad -\frac{b}{Y}-\frac{a}{X}-\frac{x}{X}+\frac{y}{Y}&=0. \end{aligned}\right\}$$

$$\left.\begin{aligned} \textit{or},\ \text{using (1)},\ \frac{y}{Y}-\frac{z}{Z}-\frac{a}{X}&=0,\\ \frac{z}{Z}-\frac{x}{X}-\frac{b}{Y}&=0,\\ \text{and}\quad \frac{x}{X}-\frac{y}{Y}-\frac{c}{Z}&=0. \end{aligned}\right\}$$

Since these three equations are not independent, as their sum vanishes identically by virtue of equation (1), any two of these three equations may be taken as the equations of the line of action of the resultant.

Ex. 8. Two forces $2P$ and P act along the lines whose equations are $y=x\tan\alpha$, $z=c$; and $y=-x\tan\alpha$, $z=-c$, respectively. Find the equations of the central axis. [C. H. 1967]

Equations of the lines in the symmetrical form are,

$$\frac{x-0}{\cos\alpha}=\frac{y-0}{\sin\alpha}=\frac{z-c}{0}\ ;\ \text{and}\ \frac{x-0}{\cos\alpha}=\frac{y-0}{-\sin\alpha}=\frac{z+c}{0}.$$

$\therefore$ We have the scheme :

$$\begin{pmatrix} 0 & 0 & c \\ 2P\cos\alpha & 2P\sin\alpha & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & -c \\ P\cos\alpha & -P\sin\alpha & 0 \end{pmatrix}$$

$\therefore\quad X=3P\cos\alpha,\ Y=P\sin\alpha,\ Z=0\ ;$

$L=-3Pc\sin\alpha,\ M=Pc\cos\alpha,\ N=0.$

$\therefore$ Equations of the central axis are,

$$\frac{-3Pc\sin\alpha+z.P\sin\alpha}{3P\cos\alpha}=\frac{Pc\cos\alpha-z.3P\cos\alpha}{P\sin\alpha}$$
$$=\frac{-x.P\sin\alpha+y3P\cos\alpha}{0}.$$

From last one, $y=(\frac{1}{3})x\tan\alpha$.

From first two, $z(\sin^2\alpha+9\cos^2\alpha)=c(3\cos^2\alpha+3\sin^2\alpha)$

$$\textit{or},\quad z=\frac{3c}{\sin^2\alpha+9\cos^2\alpha}.$$

Hence the equations of the central axis are,

$$y=\tfrac{1}{3}x\tan\alpha\qquad z=\frac{3c}{9\cos^2\alpha+\sin^2\alpha}.$$

Ex. 9. A force F acts along the axis of z, and a force mF along a straight line, intersecting the axis of x at a distance c

from the origin and parallel to the plane of yz. Show that as this straight line turns round the axis of x, the central axis of the forces generates the surface $\{m^2z^2+(m^2-1)y^2\}(c-x)^2=x^2z^2$.

The force F acts at $(0, 0, 0)$ and has components $(0, 0, F)$. The force mF acts at $(c, 0, 0)$. Its line of action is parallel to the yz-plane. Suppose it makes with the y-axis an angle θ. Then its direction cosines are $(0, \cos\theta, \sin\theta)$. $\therefore$ the components of mF are $(0, mF\cos\theta, mF\sin\theta)$

Hence we have the scheme :

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & F\end{pmatrix};\ \begin{pmatrix}c & 0 & 0\\ 0 & mF\cos\theta & mF\sin\theta\end{pmatrix}$$

$\therefore$ $X=0,\ Y=mF\cos\theta,\ Z=F+mF\sin\theta$;

$L=0,\ M=-cmF\sin\theta,\ N=cmF\cos\theta.$

$\therefore$ Equations of the central axis are,

$$\frac{-y(F+mF\sin\theta)+zmF\cos\theta}{0}=\frac{-cmF\sin\theta+x(F+mF\sin\theta)}{mF\cos\theta}$$

$$=\frac{cmF\cos\theta-xmF\cos\theta}{F+mF\sin\theta}$$

or, $-y(1+m\sin\theta)+mz\cos\theta=0$......(1)

or, $mz\cos\theta-my\sin\theta=y$..............(2)

and, $\dfrac{x+(x-c)m\sin\theta}{m\cos\theta}=\dfrac{(c-x)m\cos\theta}{1+m\sin\theta}=\dfrac{(c-x)y}{z}$ from (1)

or, $zx=(c-x)[my\cos\theta+mz\sin\theta]$

or, $mz\sin\theta+my\cos\theta=\dfrac{zx}{c-x}$......(3)

To get the locus, we eliminate θ :

Square and add (2) and (3) : $m^2z^2+m^2y^2=y^2+\dfrac{z^2x^2}{(c-x)^2}$.

$\therefore$ Locus of central axis is $[m^2z^2+(m^2-1)y^2](c-x)^2=z^2x^2$.

Ex. 10. If a system consists of two forces one of which acts along Oz and the six components of the system are X, Y, Z, L, M, N, show that the force along Oz is $\dfrac{LX+MY+NZ}{N}$.

Let P be the force which acts along Oz and let the other force act at the point (a, b, c). The components of the other force parallel to the axes are evidently $X, Y, Z-P$. Hence we have the schemes $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ X & Y & Z-P \end{pmatrix}$.

$$\therefore \quad L=b(z-P)-cY,\ M=cX-a(Z-P),\ N=aY-bX.$$

$$\therefore \quad LX+MY+NZ=-bPX+aPY=P(aY-bX)=PN$$

$$\therefore \quad P=\frac{LX+MY+NZ}{N}$$

EXAMPLES

1. Three components of forces along the axes are 1 lbs., 2 lbs., and 7 lbs. respectively, and three components of couples are 4 ft. lbs., 5 ft. lbs. and -2 ft. lbs. Find the equations to the line of action of the resultant. [C. H. 1964]

2. Equal forces P act along two perpendicular diagonals of opposite faces of a cube of side a ; find the resultant wrench.

3. Show that the following system of forces reduce to a single force :
(*i*) Force $5\sqrt{2}$ acting at $(0, 0, \frac{1}{5})$ in direction $(3, 4, -5)$, (*ii*) Force $3\sqrt{11}$ acting at $(\frac{1}{3}, 0, 0)$ in direction $(5, 5, -7)$, (*iii*) Force $\sqrt{11}$ acting at $(0, \frac{1}{4}, 0)$ in direction $(-3, 1, 1)$, axes being rectangular.

4. Three forces each equal to Q, act on a body ; one at the point $(1, 0, 0)$ parallel to Oy, the second at the point $(0, 1, 0)$ parallel to Oz, and the third at the point $(0, 0, 1)$ parallel to Ox ; the axes being rectangular, find the equations of the central axis. [C. H. 1963]

5. Forces X, Y, Z act along the three lines given by the equations $y=0, z=c$; $z=0, x=a$; $x=0, y=b$; prove that the pitch of the equivalent wrench is $\dfrac{(aYZ+bZX+cXY)}{(X^2+Y^2+Z^2)}$. If the wrench reduces to a single force, show that the line of action of the force must lie on the hyperboloid

$$(x-a)(y-b)(z-c)-xyz=0.$$

6. Two equal forces act one along each of the straight lines

$$\frac{x\mp a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{\mp b\cos\theta}=\frac{z}{c};$$

show that their central axis must, for all values of θ, lie on the surface

$$\left(\frac{x}{z}+\frac{z}{x}\right)=b\left(\frac{a}{c}+\frac{c}{a}\right).$$

7. OA, OB, OC are three co-terminous edges of a cube and AA', BB', CC', OO' are diagonals ; along BC', CA', AB' and OO' act forces equal to X, Y, Z and R respectively ; show that they are equivalent to a single resultant if $(YZ+ZX+XY)\sqrt{3}+R(X+Y+Z)=0$.

8. A single force is equivalent to component forces X, Y, and Z along the axes of co-ordinates and to couples L, M, N about these axes ; show

that the magnitude of the single force is $\sqrt{X^2+Y^2+Z^2}$ and that the equation to its line of action is $\frac{yZ-zY}{L}=\frac{zX-xZ}{M}=\frac{xY-yX}{N}=1$.

9. Two equal forces act along generators of the same system of the hyperboloid $\frac{x^2+y^2}{a^2}-\frac{z^2}{b^2}=1$, and cut the plane $z=0$ at the extremities of perpendicular diameters of the circle $x^2+y^2=a^2$: show that the pitch of the equivalent wrench is $\frac{a^2b}{a^2+2b^2}$.

10. Forces P, Q, R, P, Q, R act along the edges BC, CA, AB, AD, BD, CD of a regular tetrahedron $ABCD$. Show that they are equivalent to a wrench of pitch $\frac{a}{2\sqrt{2}}$, where a is the length of an edge.

11. Prove that if equal forces act along the edges BC, CA, AB, DA, DB, DC of a regular tetrahedron, the central axis is the perpendicular from D to the plane ABC and the pitch of the equivalent wrench is $\frac{a}{2\sqrt{2}}$, where a is an edge of the tetrahedron.

12. Forces P, Q, R act along any three mutually perpendicular generators of the same system of the surface $x^2+y^2=2(z^2+a^2)$, the positive direction of the forces being towards the same side of the plane xy. Prove that the pitch of the equivalent wrench is $\frac{2a(PQ+QR+RP)}{(P^2+Q^2+R^2)}$.

13. Two forces P, Q act along the straight lines whose equations are $y=x\tan\alpha$, $z=c$ and $y=-x\tan\alpha$, $z=-c$ respectively. Show that their central axis is $y=x\cdot\frac{P-Q}{P+Q}\tan\alpha$, $\frac{z}{c}=\frac{P^2-Q^2}{P^2+2PQ\cos 2\alpha+Q^2}$. For all values of P and Q, prove that this line is a generator of the surface

$$(x^2+y^2)\,z\sin 2\alpha=2cxy.$$

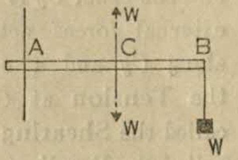
14. Any number of wrenches of the same pitch p act along generators of the same system of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. Show that they will reduce to a single resultant provided their central axis is parallel to a generator of the cone $\left(p+\frac{bc}{a}\right)x^2+\left(p+\frac{ca}{b}\right)y^2+\left(p-\frac{ab}{c}\right)z^2=0$.

15. Along the normal at every point of an octant of an ellipsoid cut off by the principal planes acts a force proportional to the element of surface at P. Show that these forces are equivalent to a single force acting along the line $a\left(x-\frac{4a}{3\pi}\right)=b\left(y-\frac{4b}{3\pi}\right)=c\left(z-\frac{4c}{3\pi}\right)$, where $2a$, $2b$, $2c$ are the axes of the ellipsoid.

BENDING MOMENTS

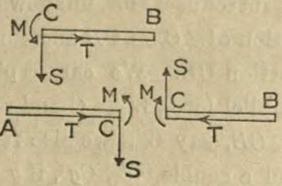
8·1. When a rod or a beam is in equilibrium under the action of external forces, internal forces, called stresses, are set up in the rod or beam. In this chapter we shall discuss about such forces which act across any imagined section of the rod or beam.

First we consider a simple case. AB is a thin horizontal rod, fixed in a wall at end A and from end B a weight W is suspended. The rod is in equilibrium and, therefore, every part of it is also in equilibrium. Consider the separate equilibrium of the part CB. It cannot be in equilibrium under the action of W only or W and the wt. of portion CB, if the rod is heavy. The other forces which act on it are the internal forces called into play across C, which constitute the action of the portion AC on portion CB. Imagine the portion CB severed from the rest by a section at C, introduce the unknown forces at C which constitute this action of AC on CB and consider the equilibrium of the severed portion CB. We can replace the wt. W at B by an equal and parallel force W at C and a couple (W, CB). If there be any wt. of CB, say W', we also replace W' similarly by a force W' at C and a couple (W', Cg), if g be the point where W' acts. The two forces combine into a single force $(W+W')$, and the two couples combine into a single couple G. These are balanced by the unknown forces acting at C. Hence these unknown forces must consist of a vertical upward force $(W+W')$ and a couple $(-G)$. These two, a force and a couple, constitute the action of AC on CB. The general case is treated in the same way. Suppose a rod or beam AB is in equilibrium under the action of a given system of external forces. To find the internal forces acting across the section at a point C of AB, we consider the separate equilibrium of CB. The internal forces across C, which constitute the action of AC on CB together with all those external forces which act on CB are in equilibrium. We take a point O of the section at C as origin and reduce the external forces acting on CB to a single force R at O and a couple G. The reversed R at O and the



reversed couple $(-G)$ taken together constitute the internal forces or stresses across C. Thus the stresses called into play across any section of the rod are equivalent to a force and a couple which are equivalent to the external forces acting on one part of the rod.

8·2. We shall henceforth confine our discussion to the simple case in which the rod is very thin and all the forces acting on the rod are in a vertical plane containing the rod. The problem becomes a two dimensional one We take the origin at C and rect. axes of which one, say Cx, is along the length of the rod and the other Cy is perpendicular to the rod. We reduce the external forces acting on CB to a force T along Cx, a force S along Cy and a couple M. The force T along the rod is called the **Tension** at C; the force S perpendicular to the rod is called the **Shearing Stress** or **Shear** at C; and the couple M is called the **Bending Moment** at C. The axis of the couple M is horizontal and perpendicular to the rod. The *bending moment* at any point *is taken as the measure of the tendency of the rod to break at that point.* It is equal to the sum of the moments of all the forces which act on one side of the point.

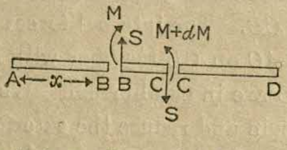


In denoting the action of AC on CB, we shall take S at C to act upwards and M to be clockwise. The action of CB on AC, being equal and opposite, will be denoted by S at C acting downwards and M acting counter-clockwise.

In the case of rods T is of little consequence and we shall be only concerned with S, the shear and M, the bending moment.

8·3. **Theorem**: *A thin straight horizontal rod rests in equilibrium. To find the shear and the bending moment at any point of the rod.*

Let AD be the rod. Take A as origin and AD as x-axis. Let B be any point of the rod where $AB=x$ and let S be the shear and M the bending moment at B. Consider the separate equilibrium of a small element BC of the rod of length dx. It is separated from the rest of the rod



by the sections at B and C. The action of the portion AB on it consists of the shear S acting at B upwards and the couple M which acts clockwise and the action of the portion CD on it consists of the shear $S+dS$ which acts at C downwards and the couple $M+dM$ which acts counter clockwise. (S and M are assumed to change continuously).

I. First we assume the rod to be *light* and no external forces to be acting on it. In this case, the element BC is in equilibrium under the action of forces S, $S+dS$ and couples M, $M+dM$. Resolving vertically and taking moments about B, we get,

$$S+dS-S=0 \quad \textit{or,} \quad dS=0 \quad \therefore \quad S=\text{constant}$$

$$\text{and} \quad M+dM-M-(S+dS)dx=0$$

$$\textit{or,} \quad \frac{dM}{dx}=S, \text{ in the limit.}=\text{constant}=a, \text{ say}$$

$$\therefore \quad M=ax+b, \text{ where } a, b \text{ are constants.} \cdots\cdots\cdots (1)$$

Thus the shear is constant, it has the same value at every point of the rod but the bending moment is variable, its value at a point is given by (1).

Let the shear or bending moment at a point B of the rod where $AB=x$, be represented by an ordinate y at B. The locus of the point (x, y) is the graph of the shear or bending moment. Here the graph of the shear is a horizontal straight line $y=a$, and the graph of the bending moment is the straight line $y=ax+b$, whose slope is constant and equal to S.

II. Secondly, suppose the rod carries a uniformly distributed load ω per unit length. The load on the element BC is ωdx acting at its middle point. This is an additional force acting on BC. Resolving vertically and taking moments about the point B, we have, $S+dS-S+\omega dx=0$ *or,* $\dfrac{dS}{dx}=-\omega$

$$\therefore \quad S=-\omega x+c, \text{ where } c \text{ is a constant ;}$$

$$\text{and} \quad M+dM-M-(S+dS)dx-\omega dx.\frac{dx}{2}=0$$

$$\textit{or,} \quad \frac{dM}{dx}=S, \text{ in the limit.} \quad \therefore \quad \frac{dM}{dx}=-\omega x+c$$

$$\text{and} \quad M=-\tfrac{1}{2}\omega x^2+cx+d, \text{ where } c, d \text{ are constants.}$$

Also $\frac{d^2M}{dx^2}=\frac{dS}{dx}=-\omega$, a constant.

In this case, the shear S and the bending moment M vary continuously with x. The graph of S is a straight line while that of M is a parabola whose axis is vertical.

It should be noted that the slope of S is $-\omega$ and that of M is S, the shear. Thus *the slope of the bending moment is equal to the shear whether the rod is loaded or not.*

III. Thirdly, suppose the rod carries a concentrated load W at a point P in the immediate neighbourhood of B so that the distance PB can be taken as $\epsilon.BC$ where $0<\epsilon<1$. Resolving vertically and taking moments about B, we have,

$$S+dS+W-S=0 \quad or \quad dS=-W.$$

This shows that dS is not a small quantity and so there is a discontinuity in the value of S as the point moves from one side of the concentrated load to the other. So we should take S' as the unknown value of the shear at C in place of $S+dS$ and M' for $M+dM$. Then above equation is $S'+W-S=0$ *or* $S'=S-W$. Thus the value of the shear decreases by an amount equal to the concentrated load in passing from the left to the right of the load. Thus *a concentrated load at a point causes a discontinuity in the shear at that point.*

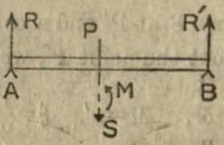
And we have, $M'-M-S'.BC-W\epsilon.BC=0$; now, we let the points B and C which are on opposite sides of the concentrated load at P, both move upto P. Then $BC\to 0$ and we have $M'=M$. Thus the value of the *bending moment does not change in passing from the left to the right of the concentrated load.* But as the slope of the bending moment is equal to the shear, which changes, it follows that there is a discontinuity in the slope of the bending moment at the point where the concentrated load is, though the graph of the bending moment is continuous at that point.

Hence if we have a thin horizontal rod in equilibrium under concentrated loads and pressures of supports at different points only, the graph of S will consist of several broken horizontal lines and that of M will consist of a continuous chain of inclined lines.

It may be observed that if different parts of a rod are under different conditions stated in I, II and III the results derived above for any particular condition will hold for that part of the rod only which satisfies that condition.

Examples Worked

Ex. 1. A uniform heavy rod AB is supported at each end. If ω be the weight per unit length, prove that the bending moment at any point P is $\frac{1}{2}\omega.AP.BP$.



Let $AB=2a$ and let R, R' be the pressures of the supports at A, B on the rod. Weight of the rod$=2a\omega$.

$\therefore\ R=R'=a\omega.$ Let $AP=x.$

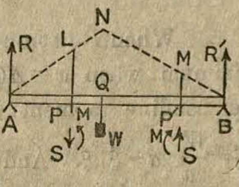
Consider the equilibrium of AP. Forces acting on it are the pressure R at A, its weight ωx acting at middle point of AP and the action of the portion PB on it which consists of a shear S acting downwards at P and a couple M acting counterclockwise. Taking moments about P, we have,

$$M+\omega x.\frac{x}{2}-R.x=0 \quad or, \quad M=a\omega x-\frac{\omega x^2}{2}=\frac{\omega x}{2}(2a-x)$$

$\therefore$ Bending moment at $P=\frac{1}{2}\omega.AP.BP$.

Ex. 2. A light rod is supported at A and B and carries a load at Q. Prove that the bending moment at P is proportional to $AP.QB$ or to $AQ.PB$ according as P lies on AQ or on QB.

[C. H. 1964, 1966]



Let $AB=2a$ and let W be the load at Q. Let $AQ=\xi$, then $BQ=2a-\xi$. Let R, R' be the pressures of the supports and A and B on the rod. Then we have, $R+R'=W$ and $R.\xi=R'(2a-\xi)$

$$\therefore\ \frac{R}{2a-\xi}=\frac{R'}{\xi}=\frac{W}{2a} \quad \therefore\ R=\frac{W}{2a}(2a-\xi) \text{ and } R'=\frac{W}{2a}.\xi \ldots\ldots(1)$$

First, let P lie on AQ and let $AP=x$. Consider equilibrium of the portion AP. Forces acting on it are the pressure R at A and the action of the portion PB on it which consists of a vertical shear S at P and a couple M acting counterclockwise. Taking moments about the point P, we have, $M-Rx=0$

or, $M = x.\frac{W}{2a}(2a-\xi) = \frac{W}{2a}.AP.BQ.$ [from (1)]

Thus the bending moment at $P \propto AP.BQ$.

Secondly, let P lie on QB, Let us denote its position by P' and let $BP' = x'$. This time consider the equilibrium of the position BP'. It will be more convenient as it does not contain the weight W. Forces acting on it are the pressure R' at B and the action of the portion AP' on it which consists of a vertical shear S at P' and a couple M acting clockwise. Taking moments about the point P', we have, $M - R'.x' = 0$

or, $M = x'.\frac{W}{2a}\xi = \frac{W}{2a}.BP'.AQ$ [from (1)]

Thus the bending moment at $P' \propto AQ.BP'$.

Hence the bending moment at P varies as $AP.QB$ or as $AQ.PB$ according as P lies on AQ or on QB.

Ex. 3. A light rod is supported at A and B. A man of weight W walks slowly along the rod from A to B. Find when and where the bending moment is maximum and find the maximum value.

As in Ex. 2 above (see its fig.), when the man is at Q where $AQ = \xi$, the bending moment at a point P, when P lies on AQ is $M = Rx$, where $AP = x$ and $R = \frac{W}{2a}(2a - \xi)$; and when P lies on BQ, the bending moment is $M = R'x'$, where $BP = x'$ and $R' = \frac{W}{2a}\xi$.

$\therefore$ When P lies on AQ, M increases as x increases and is maximum when $x = AQ = \xi$; that is, of all points of AQ, the bending moment is maximum at Q where its value is $R\xi = \frac{W}{2a}(2a-\xi).\xi$. And when P lies on BQ, M increases as x' increases and is maximum at Q when $x' = 2a - \xi$; that is of all points of BQ, the bending moment is maximum at Q where its value is $R'(2a-\xi) = \frac{W}{2a}\xi.(2a-\xi)$.

Thus when the man is at Q, the bending moment is maximum at Q and the maximum value is $\frac{W}{2a}\xi(2a-\xi)$. This also follows more easily from the graph of M which consists of two straight lines AN, BN; the maximum ordinate is at Q and is QN.

Now $\frac{W}{2a}\xi(2a-\xi)=\frac{W}{2a}[a^2-(\xi-a)^2]$, has the greatest value $\frac{W}{2a}.a^2$ when $\xi=a$, i.e., the maximum bending moment has the greatest value when the man is at the middle point of AB and the greatest value is $\frac{Wa}{2}$.

Hence when the man stands at any point of the rod, the bending moment is maximum at the point where the man stands; and when the man walks from A to B, the bending moment is maximum when he is at the middle point of the rod, the maximum value being $\frac{1}{2}Wa$.

Ex. 4. A uniform plank of length $2l$ and weight W per unit length is supported in a horizontal position at its ends. A boy whose weight is nW attempts to walk across the plank. If M is the maximum bending moment the plank will stand, show that the boy can only cross the plank if $n<\frac{2M}{lW}-l$. [C. H. 1965]

Let AB be the plank and let the boy walk from A to B. If we consider only the weight on the plank as if the boy is weightless the bending moment at a point P, distant x from A

$=\frac{W}{2}(2lx-x^2)$, [See Ex. 1. above]

$=\frac{W}{2}[l^2-(x-l)^2]$, which is maximum when $x=l$. Thus the bending moment is maximum at the middle point of AB and the maximum value is $\frac{W}{2}l^2$.

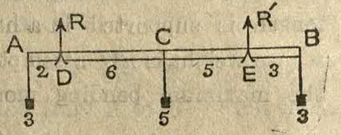
If we consider the plank AB to be weightless and the boy to be heavy and walking across it, the bending moment is maximum at the middle point of AB, when the boy is there, and the maximum value is $\frac{nWl}{2}$ [See Ex. 3. above]. Since the bending moment at any point P of the plank is the sum of the bending moments at P for the two cases (1) when the plank has weight, and the boy is weightless, (2) when the plank is weightless and the boy has weight, it follows that the bending moment is maximum at the middle point of the plank, when both the plank and the boy are considered to have weights ; and the maximum value of the bending moment is $\frac{Wl^2}{2}+\frac{nWl}{2}$. Hence in order that the boy may walk across the plank, we must have $\frac{Wl^2}{2}+\frac{nWl}{2}<M$.

$$or, \quad n<\frac{2M}{Wl}-l.$$

Ex. 5. A light horizontal rod 16 ft. long is supported at points distant 2 ft. from one end and 3 ft. from the other, and is loaded with a weight of 5 ℔ at its middle point and weights of 3 ℔ at each end. Find the shearing stress and the bending moment at any point of the rod. [M. T.]

Let AB be the rod and C its middle point. Let D and E be the supports where $AD=2$ ft. and $BE=3$ ft. and let R, R' be the pressures on the rod at D and E. There are concentrated loads of 3, 3 and 5 ℔s. at A, B and C respectively. We have $DC=6$ ft., $CE=5$ ft., $R+R'=11$, $R.6=R'.5$

$$\therefore \quad R=5, \quad R'=6.$$



Let P be any point of the rod where $AP=x$.

When $0<x<2$, P lies on AD. Consider equilibrium of the portion AP of the rod. The action of PB on AP consists of a vertical *downward* shear S at P and a *counterclockwise* couple M. Resolving vertically and taking moments about P, we have :

$$3+S=0, \text{ and } M+3x=0 \qquad \therefore \quad S=-3, \text{ and } M=-3x.$$

When $2<x<8$, P lies on DC. When P crosses D where there is a pressure R, the shear increases by R and becomes $(-3+R)=-3+5=2$; and $\frac{dM}{dx}=S=2 \quad \therefore \quad M=2x+A$.

At D, $M=-3\times 2=-6$ and M is continuous at D.

$\therefore \quad M=-6$ when $x=2 \quad \therefore \quad -6=4+A$ *or*, $A=-10$

$\therefore \quad M=2x-10 \quad \therefore \quad S=2$ and $M=2x-10$.

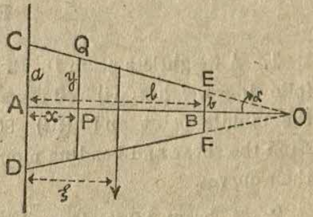
When $8<x<13$, P lies on CE. When P crosses C where there is a concentrated load of 5 ℔s., S becomes $(2-5)=-3$; and $M=-3x+B$. At C, $x=8$, $M=2.8-10=6$.

$\therefore\ M=6$ when $x=8$. $\therefore\ 6=-3.8+B\ \ \therefore\ B=30$
$\therefore\ M=-3x+30\ \ \therefore\ S=-3$ and $M=-3x+30$.
When $13<x<16$, P lies on EB. There is a pressure R' at E.
$\therefore\ S$ becomes $-3+R'=-3+6=3$, and $M=3x+C$.
At E, $M=-3.13+30=-9$. $\therefore\ M=-9$ when $x=13$
$\therefore\ -9=3.13+C\ \ \therefore\ C=-48\ \ \therefore\ M=3x-48$.
$\therefore\ S=3$ and $M=3x-48$.

Hence when $0<x<2$, $S=-3$, $M=-3x$;
when $2<x<8$, $S=2$, $M=2x-10$;
when $8<x<13$, $S=-3$, $M=-3x+30$;
and when $13<x<16$, $S=3$, $M=3x-48$.

The graph of S consists of 4 horizontal lines and that of M of 4 continuous sloping lines.

Ex. 6. A beam AB of uniform material projects horizontally from a wall at A. It is of length l and its cross section tapers uniformly from a^2 at A to b^2 at B. Prove that the greatest bending moment is $\frac{1}{12}\omega l^2(a^2+2ab+3b^2)$, where ω is the density of the material. [C. H. 1963]



We assume the cross-section of the beam to be a square. Figure shows the vertical section of the beam through A, B, the centres of the end faces of the beam which are squares of sides a and b respectively. $\therefore\ CD=a$, $EF=b$.
Let CE meet AB at O and let $\angle COA=\alpha$.

$$\therefore\quad AO=\frac{a}{2}\cot\alpha,\quad BO=\frac{b}{2}\cot\alpha$$

$$\therefore\quad \frac{a}{2}\cot\alpha-\frac{b}{2}\cot\alpha=AB=l\quad or,\quad \cot\alpha=\frac{2l}{a-b}\text{......(1)}$$

Since the bending moment at any point P is the moment of the weight of the portion PB about P, it follows that the bending moment is maximum at A. So we have to find the moment of the weight of the beam AB about A. We divide the beam into thin elementary square discs by planes perpendicular to AB. Let us find the weight of the elementary disc at P where $AP=x$; let $PQ=y$.

$$\therefore\quad y = PO\tan\alpha = \left(\frac{a}{2}\cot\alpha - x\right)\tan\alpha = \frac{a}{2} - x\tan\alpha$$

$\therefore$ Area of cross-section at $P = 4y^2 = (a-2x\tan\alpha)^2$

$\therefore$ Weight of the element at $P = (a-2x\tan\alpha)^2.dx.\omega$.

Its moment about $A = x.(a-2x\tan\alpha)^2.dx.\omega$.

$\therefore$ Greatest bending moment = bending moment at A

= sum of the moments of the weights of the elements about A

$$= \int_0^l x(a-2x\tan\alpha)^2.dx.\omega$$

$$= \omega\left[a^2.\frac{l^2}{2} - 4a\tan\alpha.\frac{l^3}{3} + l^4\tan^2\alpha\right]$$

$$= \frac{\omega l^2}{12}[6a^2 - 8a.(a-b) + 3.(a-b)^2],$$ [$\because$ $2l\tan\alpha = (a-b)$, from (1)]

$$= \frac{\omega l^2}{12}(a^2 + 2ab + 3b^2).$$ [Here ω is the weight per unit volume.]

EXAMPLES

1. A weightless beam 16 ft. long, supported at the ends, carries the following weights (*i*) 40 lbs. at a point 4 ft. from one end (*ii*) 60 lbs. at the middle point and (*iii*) 80 lbs. at a point 4 ft. from the other end. Find the shear and bending moment at any point of the rod and also draw their curves.

2. A beam 20 ft. long, is supported at one end and at a point 4 ft. from the other end. It has a uniformly distributed load of 400 lbs. per foot run and a load of 8000 lbs. at the other end. Find the maximum bending moment and the point where it occurs.

3. A beam 80 ft. long is fixed at one end and is supported at the middle point and rests horizontally. It is loaded with 5 tons at the other end. Find the curve of bending moment.

4. A beam AB is fixed at A and rests horizontally. Draw the curves of bending moment (1) when it is uniformly loaded, (2) when its weight is neglected but it supports a weight W at a given point.

5. A beam AB, 10 ft. long, is supported at two points 2 ft. and 7 ft. from A. Weights of 1 and 2 tons are placed at A and B, and in addition there is a uniformly distributed load of 2 tons per foot run between the supports. Draw the curve of bending moment and find where the bending moment is zero.

6. AB is a stiff uniform beam, of weight W and length $2l$, and is supported at its ends so as to be horizontal; a man of weight W' stands on it at P where $AP = \xi (< l)$. Show that the curve of bending moment consists of two arcs of parabolas of equal latera recta, and that the bending moment is greatest at a point distant $l - \frac{W'}{W}\xi$ from A.

7. A uniform girder is supported at its ends and the load is concentrated at its middle. Show that the maximum bending moment is twice as great as it would be if the load were uniformly distributed.

8. A uniform rod ABC of length a is smoothly hinged at A and supported at B at an inclination of 45° to the horizontal, where $AB = \frac{3}{4}a$. Find the bending moment at a point of the rod and show that it is zero at a distance $\frac{2}{3}a$ from the end A.

9. The lower end A of a thin uniform rod is attached to a smooth hinge, its upper end B resting against a smooth vertical plane; show that the bending moment at any point P varies as $AP.BP$.

10. A uniform beam of length $2l$ rests symmetrically on two supports which are at a distance $2a$ apart in a horizontal line; prove that the beam is least liable to break if $a = l(2 - \sqrt{2})$, it being given that the beam cannot bear more than a given maximum bending moment.

11. A man, of weight W, can just walk across a certain horizontal plank, of weight nW and length l, without breaking it if the plank is supported at its two ends. Show that if the plank, still horizontal, is clamped at one end and free at the other, the man can only venture along it for a distance $\frac{1}{4}l\,(1 - \frac{3}{2}n)$.

12. A light horizontal rod, of length a, whose ends are supported, is loaded so that the bending moment at any point is proportional to the load per unit length at that point. Show that the load at any point varies as $\sin \frac{\pi x}{a}$, where x is the distance of the point from an end of the rod.

13. A uniform horizontal beam AB is supported at one end A and at some other point C. If the greatest possible uniformly distributed load is to be placed upon it without breaking it, show that C must divide the beam in the ratio $\sqrt{2} + 1 : 1$.

14. A uniform rod, of length $2l$, rests symmetrically on two props in the same horizontal line at a distance $2c$ apart. If $l < 2c$, show that the tendency to break will be greatest at a prop or in the middle according as $2l \lesseqgtr (2 + \sqrt{2})\,c$; but if $l > 2c$, the tendency to break will be greatest at a prop. What happens if $2l = (2 + \sqrt{2})c$?

ANSWERS WITH HINTS

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(4) $X=2W, Y=-W$; $x+2y+a=0$.

(5) $H\sqrt{2}$; $x-y=1$. (6) $(3a, 0), (0, 3a)$.

(7) $\tan^{-1}\left(\dfrac{M_1y_2-M_2y_1}{M_1x_2-M_2x_1}\right)$;

$$\sqrt{(M_1x_2-M_2x_1)^2+(M_1y_2-M_2y_1)^2} \div |\, x_1y_2-x_2y_1 \,|.$$

(9) $(P^2+Q^2+R^2+\frac{8}{5}PR-\frac{6}{5}QR)^{\frac{1}{2}}$;

$5(Px-Qy)+R(4x+3y-5)=0$

(10) $\dfrac{P}{\sqrt{2}}(x+y-3)-\dfrac{Q}{\sqrt{5}}(2x+y-1)+\dfrac{R}{\sqrt{2}}(x-y+1)=0$.

(12) $2G$. (15) If all the forces be turned through 90°, they will meet at O', the other end of diameter through O. Since they cannot reduce to a couple, they will be in equilibrium. Hence $V=0$.

Examples p. 28

(1) $\left(\dfrac{ab^2}{\pm\sqrt{a^4+b^4}}, \dfrac{a^2b}{\pm\sqrt{a^4+b^4}}\right)$.

(4) Points determined by the eqn. $r^m=(2a-r)^n$, where $SP=r$, and major-axis$=2a$; also the end points of the major axis. (6) Of rod BC on pin at C: $\dfrac{W}{\sqrt{3}}$ along BC and $\dfrac{W}{2}$ vertically downwards; Of rod AC on pin at A: $\dfrac{2W}{\sqrt{3}}$ along AC and $\dfrac{W}{2}$ vertically downwards. (7) Take moments about A for rod AB and about C for rod BC. (8) Forces act all outwards and are proportional to the sides they bisect; if actions at B, C act along LB, LC, they are equal as $\angle BLC$ is bisected by the force at mid-point of BC which balances them; if OB, OC be ⊥^{s} on BL, CL, then sides of $\triangle BOC$ are ⊥ to the forces which keep the rod BC in equilibrium and ∴ $OB=OC$ as they are proportional to the actions at B and C; hence etc. (9) Assume strings are attached to pins which are separate from the rods.

(10) $T=2W$; Horizontal $\frac{W}{2}$. (11) Upward force on pin at $A=5W$; consider separate equilibrium of rods AB, BC and take moments about C. (12) $T=\sqrt{3}W$; at E, horizontal $\frac{W}{\sqrt{3}}$, vertical $\frac{W}{2}$; at F, horizontal $\frac{W}{\sqrt{3}}$, vertical $\frac{3W}{2}$.

Examples p. 39

(1) $a\mu^2$. (2) $\frac{(a^2-b^2)\sqrt{3}}{4ab}$. (3) If O be the centre of force and C the centre of the wire and $\angle COP=\theta$, where P is the bead in limiting equilibrium, then $\sin(\alpha\mp\lambda-2\theta)=\pm\cos\alpha\sin\lambda$, where $W=ap\tan\alpha$ and $a=$radius.

Examples p. 63

(2) The c. g. of the shell is the mid-point of the radius $\perp$ to the plane face. (5) If the string is attached to point O of the wall, OC being constant, C is on a circle whose centre is O and radius$=OC$, assumed$>a$. The resultant reaction at A acts along ACB. (7) The resultant reaction at point of contact passes through the centre. see Ex. 5. worked p. 44.

(8) $\frac{a^2-b^2}{2ab}$. (9) Let C be the centre of sphere, A its top, G the c. g. of plank, M the man of wt. W. When new point of contact is B where $\angle ACB=\theta$, then $GB=r\theta$, $BM=nr\theta$ for resultant of W at M and nW at G must act at B. Also $\theta\leqslant\lambda$.

(10) If $\theta=$inclination of plane face to horizon, then $\tan\theta\leqslant\mu$. Take moments about point of contact with plane.

(11) If $O=$centre of sphere, $C=$centre of disc, P the patricle then resultant of W at C and ω at P passes through O. If $\theta=$ inclination of OC to vertical, $\frac{\omega}{W+\omega}\cdot CP=OC\tan\theta\leqslant OC\,\mu$.

$$\therefore\ CP\leqslant\mu\left(\frac{W+\omega}{\omega}\right)\sqrt{l^2-a^2}.$$

(13) Let A, B be the pegs, C the centre of disc, G its c. g. Let $C\hat{A}Q=C\hat{A}S=\lambda_1$, $C\hat{B}Q=C\hat{B}S=\lambda_2$. Let BS, AS meet AQ, BQ at P and R. For equilibrium, the vertical through G must

pass through the quad. area $PQRS$. See Ex. 1. worked. p. 41. $\therefore$ $c <$ distance of C from vertical through P or $R = a \sin\alpha$ $-$projection of AP or BR on AB. (P nearer to A than R); use $\triangle APB$. (14) Friction per unit area $= \frac{\mu P}{\pi(a^2 - b^2)}$; take moments about axis. $\frac{2}{3}\mu P \frac{(a^2 + ab + b^2)}{a + b}$. (15) Use equns. of equilibrium. (17) Condition is $l < 2\mu c$, where $l =$ length, $2c =$ distance between handles. (21) and (22) See Ex. 6 or 7 p. 58. (23) See Ex. 9 p. 47. (24) See Ex. 10. p. 48. (26) tumbles if $4\mu\triangle > 3a^2 - b^2 + c^2$, where $\triangle =$ area of normal section ABC, where BC is on plane with B below C. (27) See Ex. 10. p. 48. If vertical through c. g. meets AT at O, consider $\triangle OAB. OA = (a - a\tan\alpha).\cos\alpha$, $\angle ABO = \theta$, $AB = 2a$, $\angle AOB = 90 - \alpha - \theta$, $\theta \leqslant \lambda$. $\therefore$ $(\cos\alpha - \sin\alpha)\cos(\theta + \alpha) = 2\sin\theta = 2\sin(\theta + \alpha - \alpha)$ etc. (28) Two cases possible (i) turning about a peg (ii) sliding at both pegs. Assume (i) to hold. Use Lami's theorem. $F = \frac{1}{3}W$. (29) If cylinder turns about point of contact A, then resultant reaction at A is W vertically upwarps forming a couple with weight W. $\therefore$ $\alpha \leqslant \lambda$ or $\tan\alpha \leqslant \mu$. If $\tan\alpha > \mu$ the cylinder slides. Use method of Ex. 2. p. 42. (30) See solution of Ex. 13 above; use its method. Here $PQRS$ is the quadrilateral, P being to extreme left. A, B are points of contact with vertical and horizontal planes. C is centre of normal section. Cylinder turns about B if G is to right of BC; slides at A, B if G is to left of the vertical through P, condition for which is $c < \frac{a(\mu^2 + \mu)}{\mu^2 + 1}$. (31) See Ex. 2 worked p. 42; or use method of Ex. 30 solved above. (32) See Ex. 8 worked p. 61. (33) Assume cube turns; condition is $\mu > \frac{a}{2l \sin\alpha}$, given. $\therefore$ turns. Suppose this happens when height of man above the ground is $x < h$. etc. (34) See Ex. 12 worked p. 50. Take moments about B. (35) Let $AB = a$ rotate about a point I of it where $AI = \xi$ and BC rotate about I' and let Q be the reaction at B. Use Ex. 34 above. Consider AB taking moments about B. AB moves alone or both AB, BC move acc. as $\xi \gtreqless a$ or $2(\sqrt{2} - 1)W' \gtreqless W$. (36) See Ex. 9 worked p. 62. When $P < \mu A$, T is zero; when $P > \mu A$, $T > 0$ and friction at A is always μA. With A as centre and radius μA draw a semi-circle and let AL perpendicular to AB meet letter at L and let tangent at L cut AC at N. etc. (37) See Ex. 13 worked p. 51. I is the intersection of the diags. or is one of the corners acc. as the quadrilateral has not or has a re-entrant angle. (38) Use method of Ex. 14 p. 52. Here from symmetry about CI (C is centre), the resultant of frictions is along CI. $\therefore$ CI is perpendicular to AB. (39) c. f. Ex. 38 above. F is perpendicular to CS. Resolve $\perp$ to CS and take moments about S; use polar equation of conic. For evolution of integrals, use

$$\int_0^\pi \frac{d\theta}{(1-e\cos\theta)^n}=\frac{1}{(1-e^2)^{n-\frac{1}{2}}}\cdot\int_0^\pi (1+e\cos\phi)^{n-1}d\phi.$$

(40) B is the centre of rotation. Take moments about B;

$$T.a\sqrt{2}=\int_0^\pi \frac{2a\mu}{\pi}W\sin\frac{\theta}{2}d\theta=\frac{4a\mu W}{\pi}$$

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(1) z=depth of c. g. system below centre of circle
$=b\cos\theta-a\operatorname{cosec}\theta$; see Ex. 1. p. 86 when $T=0$.

(2) $z=\frac{1}{2}a\cos\theta+\frac{1}{2}b\sin\theta-c\cos(90^\circ-\theta)\sin(90^\circ-\theta)$

(3) z=height of c. g. above horizontal through
$C=a\sin\theta+\tan\theta(b\operatorname{cosec}\theta-c)$ etc.

(5) z=height of c. g. above $BC=2a\cos\frac{\theta}{2}-\frac{1}{2}\left(l-2a\sin\frac{\theta}{2}\right)$,
where $\theta=\widehat{BOC}$, l=length of the string.

(6) Use equation y^2-4ax. (7) Z=depth of c. g. below point of suspension$=l\cos\theta+a\cos\phi+\frac{3}{8}a\sin\phi$.

Also $a=l\sin\theta+a\sin\phi$, (8) See Ex. 2, worked p. 82.

(9) $AC=a\cos C+c\cos A$ and $a\sin C=c\sin A$.

(10) $T=\frac{5}{2}\omega$. (13) $T=4W$. (14) Sum of squares of diagonals=sum of squares of sides. Second part :

$$T_1 : T_2=l : l',\quad 2l^2+2l'^2=a^2+b^2 \text{ etc.}$$

(15) 'Tension of the rod' should be 'tension of the string'. AB=fixed horizontal rod, BD=string whose middle point is O. Let $\widehat{ABO}=\theta$. In position of equilibrium $a=2b\cos\theta$ etc.

(16) AOC is one straight line and AC, BD bisect each other at right angles. $\therefore$ $OA.OC=a^2-b^2$.

(18) Let $\overset{\wedge}{ABC}=\theta$ and $P.AB=L$. $T=\frac{4l}{BD}$

(20) Let $BC=b$. Equation is $W.\delta(\frac{1}{2}b\sin\alpha)-T\delta(AC)=0$ and $AC^2=a^2+b^2-2ab\cos\alpha$ etc.

(22) Let $AB=a$; $AC^2+BD^2=4a^2$, $BD^2+a^2=\frac{a^2}{2}+2.BE^2.(CE=ED)$ $\therefore$ $AC^2+2BE^2=$constant.

(25) Let $AB=a$, $BC=b$, $\overset{\wedge}{BCA}=\phi$; $AC=a\cos\theta+b\cos\phi$ etc. Also $a\sin\theta=b\sin\phi$ and in position of equilibrium $\phi=90^\circ-\theta$.

(26) $T=\frac{\omega}{2\pi}\cot\alpha$. (27) $T=\frac{W}{4\pi}\cot\alpha$.

(29) See Ex. 8 worked p. 92.

(31) See Ex. 9 worked p. 103. Remove rod EF. $Y=\frac{W}{2}$, $X=\frac{W}{\sqrt{3}}$

(33) Lower slant beams are also inclined at θ to horizon.

(34) See Ex. 9 worked p. 103. $T=4W$, remove rod AB. $X=\frac{5W}{2\sqrt{2}}$, $Y=\frac{3W}{2\sqrt{2}}$; $R=\frac{W}{2}\sqrt{17}$ at $\tan^{-1}\frac{3}{5}$ with BC.

(35) Let $PQ=2c$, show $c=\frac{a}{2\sqrt{2}}$. $\therefore$ $AP=\frac{a}{2}$. For actions at A, B remove rod AB. Remove pegs P, Q, applying $R=2\sqrt{2}W$ at the points of contact. For action at C remove rod BC. Proceed as in Ex. 9 p. 103. Actions are : at A, $\frac{3W}{2}$ horizontal ; at C, $\frac{W}{2}$ horizontal ; at B or D, $\frac{W}{2}\sqrt{5}$ at $\tan^{-1}\frac{1}{2}$ to vertical.

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(3) Let $\overset{\wedge}{ADE}=\theta$, $\overset{\wedge}{AED}=\phi$, $DE=d$,

then $\frac{\sin\theta}{q}=\frac{\sin\phi}{p}=\frac{\sin A}{d}$. Also $d^2=p^2+q^2-2pq\cos A$.

(5) c. f. Ex. 14 of previous examples solved above.

(6) Let θ, ϕ be inclinations to horizontal of radii to points where beams rest ; let w_1, w_2 be weights of beams whose c. g.'s are at points h_1, h_2 above their end points. Distance between beams=constant. $w_1\cot\theta=w_2\cot\phi$. (7) In case of two degrees

of freedom : Let AB, AC be rods, O centre of circle, $\widehat{BAC}=\theta$, ϕ = inclination of AG to vertical. z = depth below O of c. g. $=(l\cos\theta-a\operatorname{cosec}\theta)\cos\phi$. $\delta z=0$ $\therefore$ $\phi=0$, $l=\operatorname{cosec}^3\theta\cot\theta$.

(9) Take axes of ellipse as x, y axes, x axis being upwards ; let A be $(a\cos\phi,\ b\sin\phi)$ then $b\sin\phi=l\sin\theta$.

$$z=\text{c. g. above origin}=a\cos\phi-\frac{l}{2}\cos\theta \text{ etc.}$$

(10) If K = mid. point of CD, θ angle BC makes with AB produced, then $AK^2=\left(a+\frac{a}{2}\cos\theta\right)^2+\frac{9a^2}{4}\sin^2\theta$ etc.

$$T=W\sqrt{39}\ (\text{not } \tfrac{1}{2}W\sqrt{39}).$$

(12) Take O, middle point of AD a origin, OA as x-axis. Let $\widehat{OAB}=\theta$, $AB=2a$. Resolve the forces parallel to the axes and find the work done in displacement in which DA increases, O remaining its middle point and θ changes to $\theta+\delta\theta$.

A thrust $=\dfrac{P}{\sqrt{3}}$.

(13) Take horizontal through C as initial line.

(14) O origin, vertical through O initial line.

(16) If θ = inclination of AC to vertical, then $\tan\theta=\dfrac{Q-P}{Q+P}$. For tension T in AC let length of AC be slightly altered, A remaining fixed. This alters the angles of traingles ABC, ADC. Equation is $P\delta\cos(\psi+\theta)+Q\delta\cos(\psi-\theta)-T.\delta(2a\cos\psi)=0$ $(\psi=\widehat{BAC})$. As θ is constant, $T=\dfrac{\sqrt{2}PQ}{\sqrt{P^2+Q^2}}$

(17) Let $\widehat{QPR}=\theta$, T = thrust in LM, then $T=\dfrac{19}{4\sqrt{3}}W$, as $\theta=30^\circ$. T is the horizontal component of the action of hinge at L on rod LM. If Y = vertical component, then for equilibrium of LM, $Y=\dfrac{W}{4}$. $\therefore$ Action at $L=\dfrac{W}{2}\sqrt{30\tfrac{1}{3}}$.

Examples A. p. 148

(2) $h\sqrt{2}<a$, where h = height of cylindrical part only.

(3) Stable in both cases. (4) w = weight of hemisphere, kw = weight placed at centre of plane face. Stable if $k<\frac{1}{8}$.

(5) Stable if $\frac{1}{h} > \frac{1}{r} - \frac{1}{2r} = \frac{1}{2r}$, which always holds.

(7) Greatest weight $= \frac{1}{3}$ (weight of $\triangle ABC$). (9) $A =$ top of fixed sphere, $C =$ centre of hemisphere and g its c. g so that $Cg = \frac{r}{2}$. If $P =$ particle then for equilibrium AC and Pg intersect in G, the c. g. of hemisphere and particle. For stability, $CG > \frac{r^2}{R+r}$. This holds always if $\frac{r^2}{R+r} <$ perpendicular from C on $Pg = r \cdot \frac{1}{\sqrt{5}}$. (11) $A =$ top of sphere, $A' =$ new point of contact when cube has rolled through θ. If $D =$ initial point of contact of cube with sphere, $A'D = r\theta$, $DG = \frac{\pi r}{4}$. If vertical through G meets base of cube at L, then cube can rock through θ if $DL \leqslant DA'$ or $\frac{\pi r}{4} \tan\theta \leqslant r\theta$ or $\frac{\tan\theta}{\theta} \leqslant \frac{4}{\pi}$. Now $\frac{\tan\theta}{\theta}$ increases with θ in $\left(0, \frac{\pi}{2}\right)$ $\therefore$ $\frac{\tan\theta}{\theta} \leqslant \frac{4}{\pi}$ in $\left(0 \leqslant \theta \leqslant \frac{\pi}{4}\right)$. Again $AA' = r\theta \leqslant \frac{\pi r}{4}$, hence the result. (14) ρ at vertices are $4a$, $4b$. (15) Let wire be in equilibrium with $B(at^2, 2at)$ as point of contact. Then normal at B, $y + tx = 2at + t^3$ passes through G $(h, 0)$. ρ at $B = 2a(1+t^2)^{\frac{3}{2}}$, stable in $\rho > h$ etc. (16) Consider vertical section through G, c. g. of plank perpendicular to axis of cylinder. If rolled through θ, height of G above O, centre of circle $=$ projection on vertical of $(OB + BA' + A'G) = a\cos\theta + a\theta\sin\theta + h\cos\theta$. Stable if $h < a$.

(17) See Ex. 6 worked p. 126. $-dV = \left(\mu r^5 - \lambda \cdot \frac{r-a}{a}\right) dr$, where $a =$ natural length. $\lambda(r-a) = a\mu r^5$ gives positions of equilibrium.

Examples B. p. 150

(1) $\sin^3\theta = \frac{c}{a}$, ($\theta =$ inclination of rod to vertical) ; unstable.

(2) If $\theta = \widehat{CAB}$, $\theta = 60°$. (3) If $\angle POW_2 = \theta$, $\sin\theta = \frac{W_2}{2W_1}$.

(4) Let $\alpha =$ inclination of plane, $h =$ perpendicular distance AM of pulley A from plane, $\theta = \angle MAW$, $l =$ length of string, if

z = depth of c. g. below A, then $(W+P)z=Wh[\cos\alpha+\tan\theta\sin\alpha]$ $-P(l-h\sec\theta)$. $\sin\theta=\frac{W}{P}\sin\alpha$.

(5) If θ = inclination of upper side of square to vertical, then for equilibrium $\theta=30^\circ$ or 90°. First position is unstable, second stable. (7) See Ex. 9 worked p. 130. Stable.

(8) If θ = inclination of AB to horizontal, depth of C $=4a\cos\theta+a\sin\theta$. Vertical at C lies within $\angle ACB$. $\therefore\ \tan\theta<\frac{3}{4}$.

(10) Let P be $(a\cos\theta, b\sin\theta)$, Q be $(a\cos\phi, b\sin\phi)$ where $\phi>\theta$. Let $PQ=c$, z = height of c. g. above PQ.

Then $z=\frac{ab}{c}\sin(\phi-\theta)$

Also $c^2=a^2(\cos\theta-\cos\phi)^2+b^2(\sin\theta-\sin\phi)^2$. For equilibrium $\phi-\theta=\frac{\pi}{2}$ or $\phi+\theta=0$ or π, When $\phi-\theta=\frac{\pi}{2}$.

$\sin 2\theta=\frac{c^2-a^2-b^2}{a^2-b^2}$, θ real if $|\sin 2\theta|<1$ *i.e.* $b\sqrt{2}<c<a\sqrt{2}$. Unsymmetrical positions where CP, CQ are conj. diameters, unstable.

When $\phi=-\theta$, we have $\theta=-\alpha$ where $\sin\alpha=\frac{c}{2b}$, $0<\alpha<\frac{\pi}{2}$. PQ parallel to minor axis. Stable if $c>b\sqrt{2}$. When $\phi=\pi-\theta$, we have $\theta=\beta$ where $\cos\beta=\frac{c}{2a}$, $0<\beta<\frac{\pi}{2}$. PQ parallel to major axis. Stable if $c<a\sqrt{2}$.

(11) If θ = inclination to horizontal of the $\perp$ from vertex on base, z = height of c. g. above line of pegs, then

$z=h\sin\theta-\frac{2c}{\sin 2\alpha}\cdot\sin(\theta-\alpha)\sin(\theta+\alpha)$; for positions of equilibrium $\theta=\frac{\pi}{2}$ *or*, $\sin\theta=\frac{h\sin 2\alpha}{4c}$. For θ to be real $\sin\theta<1$; also $\theta>\alpha$; vertical position unstable in both cases.

(12) For equilibrium, $\theta=0$, *or* $\cos\theta=\frac{k}{2}$; θ is real if $k<2$. When $k<2$, the position $\theta=0$ is stable. All other positions are unstable. (13) See Ex. 15 worked p. 138. (14) See Ex. 19 worked p. 144. (15) See Ex. 18 worked p. 143.

Examples p. 158

(1) The hyperbola $xy=\frac{2k}{9 \sin \omega}$, where k=constant area, and $\omega=\angle xoy$. (2) Let ABC be a triangle whose base BC is fixed and vertical angle $BAC=\alpha$, given. Join AD where D is mid-point of BC. Let G be the c. g. of $\triangle ABC$. Then $DG : GA=1 : 2$. Through G draw $GE \parallel$ to AB and $GF \parallel$ to AC meeting BC in E, F; join GE, GF. Then $DE : EB=1 : 2$, $DF : FC=1 : 2$. $\therefore$ E, F are fixed points and $\angle EGF=\angle BAC=\alpha$=constant. $\therefore$ Locus of G is an arc of the circle on EF containing the angle α. (3) For G, the c g. of the vessel and water, can be lowered so long as G is above water surface. (4) If A, B, C, D be the four points, take A as origin and AB, AC, AD as axes of x, y, z. (5) Let ρ_1, ρ_2, ρ_3 be the densities of rods BC, CA, AB; their masses $a\rho_1$, $b\rho_2$, $c\rho_3$ may be taken at their mid-points.

In 1st case $R \cos A=\frac{b\rho_2+c\rho_3}{2a\rho_1}\cdot\frac{\Delta}{a}$ etc

$$\therefore \quad \frac{b\rho_2+c\rho_3}{\sin 2A}=\frac{c\rho_3+a\rho_1}{\sin 2B}=\frac{a\rho_1+b\rho_2}{\sin 2C}$$

$$=\frac{2a\rho_1}{\sin 2B+\sin 2C-\sin 2A}=\text{etc.}$$

$$\therefore \quad \frac{\rho_1}{\cos B \cos C}=\frac{\rho_2}{\cos C \cos A}=\frac{\rho_3}{\cos A \cos B},$$ hence the result.

Similarly the 2nd case. (6) See Art. 5·5 The wts. at A and $C\equiv\frac{1}{3}(W_1+W_2)$ at X and the other two wts. equal to this, may be taken to act at O and Y because BD and OY have same middle point. (7) Replace each triangle by equal wts. at its vertices. Since BD is bisected by AC result follows. Conversely if $\triangle ABC=W_1$ and $\triangle ADC=W_2$ and O is the point of intersection of AC, BD, since the c. g. is on AC. $\therefore$ $W_1.BO=W_2.DO$ but $W_1 : W_2=\triangle ABC : \triangle ADC=BO : DO$ $\therefore$ $BO=DO$ and $\therefore$ $\triangle ABC=\triangle ADC$. (8) Use result of art. 5·5; it follows that the c. g. is the intersection of the diagonals. Now show that the diagonals bisect. (11) r=in-radius$=\frac{\Delta}{s}$; use remainder$\equiv$triangle$+$($-$incircle). (12) Let g_1, g_2, g_3, g_4 be the c. g.'s of the faces opposite to vertices A, B, C, D of the tetrahedron and let the areas of the faces be a, b, c, d. Let $p=\perp$ from A on $\triangle BCD$ and $p'=\perp$ from g_1 on $\triangle g_2g_3g_4$ which is ∥ to $\triangle BCD$ then $p'=\frac{1}{3}p$. If G=c. g. reqd. then G is the c. g. of wts. a, b, c, d at $g_1\ g_2, g_3, g_4$. ∴ Distance of G from plane $g_2g_3g_4=\frac{ap'}{\Sigma a}=\frac{1}{3}\cdot\frac{ap}{\Sigma a}=\frac{\text{vol. of } ABCD}{\Sigma a}$=constant ∴ Result follows. (13) The c. g. of the cone coincides with the centre of the sphere. (14) Height of hollow cone$=\frac{1}{3}$. ht. of cone. (15) When x=ht. of liquid, $z=\frac{x^2 s\rho g+2Wb}{2(xs\rho g+W)}$. Find x for z to be min. (16) Let O=centre of circle, AB, AC the tangents, G=c. g. of stated portion which is (quad. $OBAC$ − sector OBC). If g=c. g. of quad. $OBAC$, we have $Og=\frac{a}{3}(\sec\alpha+\cos\alpha)$.

$$\therefore\quad OG=\frac{a^2\tan\alpha.\ \frac{a}{3}(\sec\alpha+\cos\alpha)-a^2\alpha.\ \frac{2}{3}a\,\frac{\sin\alpha}{\alpha}}{a^2\tan\alpha-a^2\alpha}$$

$$=\frac{a}{3}\cdot\frac{\sin\alpha\cdot\tan^2\alpha}{(\tan\alpha-\alpha)}.$$

Examples p. 163

(1) $\bar{x}=\frac{a}{4k}\cdot(4\sqrt{2}-k)$, $\bar{y}=\frac{4a}{3k}\cdot(2\sqrt{2}-1)$,

where $k=\sqrt{2}+\log(1+\sqrt{2})$.

(2) $\left(\frac{2a}{5}, \frac{2a}{5}\right)$. (4) $\left(\frac{4a}{5}, \frac{4a}{5}\right)$. (5) The eqn. is $y=(\sqrt{2}-1)x$.

(6) $(\pi-\frac{4}{3}a, \frac{2}{3}a)$.

(8) $\bar{x}=\frac{2a}{\alpha^2}(\alpha\sin\alpha+\cos\alpha-1)$, $\bar{y}=\frac{2a}{\alpha^2}(\sin\alpha-\alpha\cos\alpha)$ where α=angle subtended by arc at centre, a=radius.

Examples p. 170

(1) (*a*) $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$. (*b*) $\frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{256}{315\pi}$. (*c*) $\left(\frac{a}{5}, \frac{b}{5}\right)$ (*d*) On the conjugate diameter at a distance $\frac{4\alpha}{3\pi}$ from centre, 2α being the length of this conjugate diameter. (*e*) On diameter conjugate to given line at a distance $\frac{3h}{5}$ from vertex of the diameter, where h is the portion of the diameter intercepted between the vertex

and the line. (*f*) $\left(\frac{5a}{3}, 0\right)$. (*g*) $\sinh\frac{a}{c}\cdot\bar{x}=a\sinh\frac{a}{c}-c\cosh\frac{a}{c}+c$, $\sinh\frac{a}{c}\,\bar{y}=\frac{1}{8}\left[2a+c\sinh\frac{2a}{c}\right]$.

(2) (*a*) $\left(\frac{5a}{6}, \frac{8a}{9}\right)$ (*b*) $\bar{x}=\frac{9\pi^2-16}{18\pi}\cdot a$, $\bar{y}=\frac{7a}{6}$.

(*c*) $\bar{x}=\frac{a\pi}{4\sqrt{2}}$, $\bar{y}=\frac{3\log(\sqrt{2}+1)-\sqrt{2}}{6\sqrt{2}}.a$.

(*d*) $\left(\frac{128\sqrt{2}}{105\pi}\cdot a, 0\right)$.

(*e*) Distance from centre$=\frac{2}{3}\cdot\frac{a\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}$, whete $2\alpha=$angle of the segment. (3) $\bar{x}=\frac{a^2+ab+b^2}{a+b}$, $\bar{y}=\frac{4}{3\pi}\cdot\bar{x}$.

(4) $\bar{x}=\frac{3\pi^2-8}{4\pi}\cdot a$, $\bar{y}=\frac{5}{4}a$ (origin at vertex).

(6) Put $x=aX^{\frac{3}{2}}$, $y=bY^{\frac{3}{2}}$ and use Dirichlet's Integral

$$\int\!\!\int X^{l-1}Y^{m-1}dXdY=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}$$

where $X\geqslant 0$, $Y\geqslant 0$, $X+Y\leqslant 1$. If $\rho=kx^ly^m$ then

$$\frac{\bar{x}}{a}\cdot\frac{\Gamma\left(\frac{l+1}{n}\right)}{\Gamma\left(\frac{l+2}{n}\right)}=\frac{\bar{y}}{b}\cdot\frac{\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(\frac{m+2}{n}\right)}=\frac{\Gamma\left(\frac{l+m+n+2}{n}\right)}{\Gamma\left(\frac{l+m+n+3}{n}\right)},\text{ where } n=\tfrac{2}{3}.$$

Examples p. 176

(1) (*a*) (i) On y-axis, distance from origin$=\frac{2a}{15}\cdot\frac{15\pi-8}{3\pi-4}$.

(ii) Origin. (*b*) On axis, distance from pole$=\frac{50}{63}a$.

(*c*) On axis, distance from origin$=\frac{a}{6}(2+\sqrt{2})$.

(3) On minor-axis, distance from centre

$$=\frac{2b(a^3-b^3)}{3.\sqrt{a^2-b^2}}\div\left[a\sqrt{a^2+b^2}+b^2\log\frac{a+\sqrt{a^2-b^2}}{b}\right]$$

(5) (*a*) (i) $\bar{y}=\frac{a}{6}\cdot\frac{63\pi^2-64}{9\pi^2-16}$ (ii) Origin. (*b*) $\bar{x}=\frac{4a}{5}$

(*c*) (i) $\bar{x}=\frac{3}{8}a$, $\bar{y}=0$, $\bar{z}=0$ (ii) $(\frac{8}{15}a, 0, 0)$, where Oy is minor axis of ellipse and yOz is the plane of section.

(6) $OG=\frac{3}{4}\cdot\frac{(a+c)^2}{2a+c}$, on radius ⊥ to plane, where $a=$radius.

(7) On radius ⊥ to tangent plane, distant $\frac{a}{8}$. from centre where a=radius.

Examples p. 186

(1) (*a*) $\left(-\frac{la^2}{4}, -\frac{mb^2}{4}, \frac{a^2l^2+b^2m^2+4}{8n}\right)$

(*b*) $\bar{x}=\frac{a}{8\pi}[7\sqrt{2}+3\log(1+\sqrt{2})]$, $\bar{y}=0$ $\bar{z}=0$.

(*c*) $\left[\frac{5a}{8}, 0\ \frac{5a}{16}(m+n)\right]$

(2) Make the substitution $x=aX^{\frac{1}{2}}$, $y=bY^{\frac{1}{2}}$, $z=cZ^{\frac{1}{2}}$ and use Dirichlet's Integrals.

(1, $\left(\frac{3a}{8}, \frac{3b}{8}, \frac{3c}{8}\right)$, (2) $\frac{\bar{x}}{a}=\frac{\Gamma\left(\frac{l+2}{2}\right)\Gamma\left(\frac{l+m+n+5}{2}\right)}{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(\frac{l+m+n+6}{2}\right)}$, etc.

(3) $\bar{x}=\frac{5a}{16}\cdot\frac{2a^2+b^2+c^2}{a^2+b^2+c^2}$, etc.

(3) If the the diameter through the point be taken as x-axis and the ⊥ to the plane face at the centre the z-axis, then the c.g. is $\left(\frac{a}{7}, 0, \frac{128a}{105\pi}\right)$

(4) $\frac{\bar{x}}{a}=\frac{\bar{y}}{b}=\frac{\bar{z}}{c}=\frac{3}{4}\cdot\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{4}{n}\right)}$. (use Dirichlet's Integrals).

(5) Vol$=\pi a^2.2\pi c$, surface area$=2\pi a.2\pi c$.

Miscellaneous Examples p. 200

(1) On axis, distance from cusp$=\frac{55}{42}a$.

(2) $\bar{x}=\frac{6a}{5}$, $\bar{y}=\frac{9}{20}a$ where O is origin and OA x-axis.

Examples p. 214

(15) Parameter c is given by the eqn. $e^{\frac{2a}{c}}=\frac{b^2}{c^2}-1$.

(17) $\mu[l+\mu h-\sqrt{(1+\mu^2)h^2+2\mu hl}]$. (19) $100(\sqrt{43}-1)$ ft.

Examples p. 227

(1) $\frac{1}{4}$ ft. (6) 33·66 tons wt, 31·25 tons wt.

(10) $12T_0y=ax^4+6bx^2$. (11) $\omega \propto \cos\psi$ where ψ=inclination of tangent to horizontal. i.e. $\omega \propto \frac{1}{\sqrt{h}}$ where h=height above directrix.

Miscellaneous Examples p. 238

(3) If C be the vertex, let $CP=s_1$, $CQ=s_2<s_1$, s_1, s_2 are on opposite side of C. Given $\psi_1+\psi_2=90^\circ$ $\therefore$ $s_1+s_2=\frac{2c}{\sin 2\psi_1}$. If $QR=RP$, then $CR=\frac{s_1-s_2}{2}=-c\cot 2\psi_1$. If ϕ=inclination of tangent at R, we have $c\tan\phi=-c\cot 2\psi$ $\therefore$ $\phi=2\psi-90^\circ$ $\therefore$ $\cos\phi=\sin 2\psi$ etc. (6) For equilibrim of particle, resolving horizontally $\omega c_1=\omega c_2$ $\therefore$ $c_1=c_2$. (7) Consider equilibrium of particle alone, and then equilibrium of one ring and attached half of string only ; we get $s_0=nl$ and $c=\mu l(n+1)$. Use formula $x=c\log\frac{s+\sqrt{c^2+s^2}}{c}$. (8) Tension at B is along CB. $\therefore$ $BCw \sin\alpha=T=wy=wc\sec\alpha$. $\therefore$ height of B above directrix$=BC\sin\alpha$ etc., $s_0=c\tan\alpha$, $s_0+l-BC=c\tan\psi$, etc.

(9) $4l\cos^2\alpha>3l$.

(10) Use formulae $s+y=ce^{\frac{x}{c}}$ and $y=c\cosh\frac{x}{c}$; origin is at $(0, l)$ referred to horizontal & vertical lines through fixed ring. (11) From equilibrium of ring $s_0=\frac{b}{2}$. Use formula $s+y=ce^{\frac{a}{c}}$ for ring and one peg. (12) Depth of c.g. of the portion between pegs$=\frac{y}{2}-\frac{ac}{2s}$ where y=height of pegs above directrix, $2s$=length of this portion. See Ex. 9. p. 212.

(13) We have $l=2c\sinh\frac{a}{c}+c\cosh\frac{a}{c}$; put $\frac{a}{c}=\xi$, then $\frac{2l}{a}=\frac{3e^{\xi}-e^{-\xi}}{\xi}$. Find minimum l. (14) See hint in Ex. 12 above. Use formula, work done$=W.h$. (15) $mlw=T_0=cw$ $\therefore$ $c=ml$. From equilibrium of bar, $s_0=nl$. $\therefore$ $n=m\sinh\frac{x_0}{c}$, $n+1=m\sinh\frac{x_0+a}{c}$. Eliminate x_0

Examples p. 258

(3) A=fixed end of string, BC=portion in contact with circle; circle of centre O touches wall at D. ∴ $B\hat{O}C=\theta$, T=tension of string$=Pe^{\mu\theta}$. Consider equilibrium of the whole system and take moments about D. (4) $=We^{\mu\pi}$, $W=P'e^{\mu\pi}$, $R=P+W$, $R'=P'+W$, etc. (5) If T_1, T_2, T be tensions of AC, BC, AB ($AC<BC$) then $T_1=Te^{\mu(\pi-A)}=T_2e^{\mu(\pi-B)}e^{\mu(\pi-A)}$ or $T_1=T_2e^{\frac{3\mu\pi}{2}}$. Apply Lami's Theorem. (6) Let α, β be inclinations of BA, BC to vertical at B and μ_1, μ_2, μ_3 coeffs. of friction of pegs A, B, C. Then $P=e^{\mu_1\alpha}e^{\mu_2(\pi-\alpha-\beta)}e^{\mu_3\beta}.W$ etc. (7) Upper two wts. move up, lower wts. move down. (8) Let $ABCDE$ be the string where AB and $DE>AB$ are the hanging parts. If T=tension at B or D, then $Te^{\mu(\pi-\theta)}=w.DE$, $T=e^{\mu(\pi-\theta)}w.AB$, $T=wc\operatorname{cosec}\theta$, $\frac{a}{2}=c\log\cot\frac{\theta}{2}$ etc.

(13) $T=A+wy$, $R=\dfrac{A+wy}{\rho}-w\cos\psi=\dfrac{A+2aw-6aw\cos^2\psi}{4a\cos\psi}$. R is min. when $\psi=0$. ∴ $R=0$ when $\psi=0$ etc. Tension at cusp$=6aw$. (14) For converse, $\rho\cos\psi=A+y$; transform to cartesian diff. eqn. and solve: $y+A=c\cosh\dfrac{x+B}{c}$.

(18) $\dfrac{dT}{d\psi}+\mu T+4aw(\mu\cos^2\psi+\sin\psi\cos\psi)=0$ when $\psi=0$, $T=F$, required force; when $\psi=\frac{\pi}{2}$, $T=0$.

(19) $F=w[y-ce^{\mp\mu\psi}]$ acc. as string tends to move up or down. (20) Take origin at centre; equation of equilibrium is $\dfrac{dT}{d\theta}-\mu T=aF\left(\mu\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}\right)$; $T=0$ both when $\theta=0$ and $\theta=\pi$.

(21) F varies as (a) r^{-4}, (b) $r^{-\frac{5}{2}}$, (c) r^{-2}, (d) (i) r^{-3}, (ii) $r(r^2+a^2-c^2)^{-2}$, where c is the distance of the point from centre.

(23) $F\propto\dfrac{r}{b'}$, where b'=semi-conjugate diameter.

(24) Let F = force per unit mass, m = mass per unit length, given $m=KT$ and curve is equiangular ; to find F. We have,

$$\frac{dT}{dr}=-KTF,\ Tp=c,\ p=r\sin\alpha.$$

$$\therefore\ \frac{d}{dr}\left(\frac{c}{r\sin\alpha}\right)=-KF\cdot\frac{c}{r\sin\alpha}\quad or,\quad F=\frac{1}{Kr}\propto\frac{1}{r}.$$

(25) We get, $\frac{\mu\sigma}{2r^2}=\frac{c}{p}$ when $r=b$, $p=\frac{b}{\sqrt{2}}$ etc.

Examples p. 272

(3) $ABCD$ = tetrahedron, $\alpha, \beta, \gamma, \delta$ areas of faces opposite A, B, C, D. P, Q, R, S are the forces at A, B, C, D. Then $P=k\alpha$ etc. Let L be any line and let (P, L) denote angle between P and L, etc. Algebraic sum of the projections of the faces of $ABCD$ on plane π perpendicular to $L=0$. $\therefore$ $\Sigma\alpha\cos(P, L)=0$. $\therefore$ $\Sigma P\cos(P, L)=0$ *i.e.* sum of resolved parts of P, Q, R, S along $L=0$ and L is any line. $\therefore$ Resultant force $=0$. Again draw AM perpendicular to BCD and MN perpendicular to CD. Then AN is perpendicular to CD and $\angle ANM=\theta$ is the angle between faces α, β. Moment of P about $CD=k\alpha.MN=k\alpha.AM\cot\theta=3kV\cot\theta$, where V is vol. of $ABCD$. Similarly moment of Q about $CD=-3kV\cot\theta$. $\therefore$ Sum of moments of P, Q, R, S about edge $CD=0$, This is true for any edge. $\therefore$ Resultant couple $=0$. Hence the forces are in equilibrium.

(4). To show resultant couple is zero replace each force by three equal forces at the vertices of the face to which the force is perpendicular and take moments about the edges.

(5) $R=0$; sum of moments about every generator of opposite system is zero. $\therefore$ $G=0$.

(7) Let θ = inclination to vertical of an inclined rod. Equation of virtual work is $-W\delta(a\cos\theta-b\cot\theta)+3T.\delta.(a\sqrt{3}\sin\theta)=0$. In equilibrium $a\sqrt{3}\sin\theta=a$ etc.

(9) Let θ = semi-vertical angle of cone. Height of c g. of cone $=\frac{1}{3}h\cos\theta$, and of c. g. of ring $=h\cos\theta-r\cot\theta$ etc.

(10) See Ex. 8 worked p. 102.

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(1) $LX+MY+NZ=4+10-14=0$ $\therefore$ Single resultant : equations are $7x+5=z=\frac{1}{2}(7y-4)$.

(2) $R=P\sqrt{2}$, $\Gamma=\frac{aP}{\sqrt{2}}$; axis is $x=y=\frac{a}{2}$. (4) $x=y=z$.

(9) Scheme: $\begin{pmatrix} a\cos\theta & a\sin\theta & 0 \\ Pka\sin\theta & -Pka\cos\theta & Pkb \end{pmatrix}$,

$\begin{pmatrix} -a\sin\theta & a\cos\theta & 0 \\ Pka\cos\theta & Pka\sin\theta & Pkb \end{pmatrix}$ where $k^2=a^2+b^2$.

(10) Take the tetrahedron as being formed by the six diagonals of the faces of a cube.

(12) Generators meet plane $z=0$ at points

$(a\sqrt{2}\cos t, a\sqrt{2}\sin t, 0)$ where $t=\theta, \theta+\frac{2\pi}{3}, \theta+\frac{4\pi}{3}$.

$$X=k\sqrt{2}\left[\left(P-\frac{Q+R}{2}\right)\sin\theta+\frac{\sqrt{3}}{2}\left(Q-R\right)\cos\theta\right]$$

$$Y=-k\sqrt{2}\left[\left(P-\frac{Q+R}{2}\right)\cos\theta-\frac{\sqrt{3}}{2}\left(Q-R\right)\sin\theta\right],$$

$Z=k(P+Q+R)$, $L=aX$, $M=aY$, $N=-2aZ$.

(14) $X=ak\Sigma R_1\sin\theta_1$, $Y=-bk\Sigma R_1\cos\theta_1$, $Z=ck\Sigma R_1$,

$L=bck\Sigma R_1\sin\theta_1+pX=\left(\frac{bc}{a}+p\right)X$ etc.

(15) The force at (x, y, z) has components $kdydz$, $kdzdx$, $kdxdy$ parallel to the axes.

Examples p. 296

(1) See Ex. 5 worked p. 294; $0<x<4$: $S=80$, $M=80x$; $4<x<8$: $S=40$, $M=40(x+4)$; $8<x<12$: $S=-20$ $M=-20(x-32)$; $12<x<16$: $S=-100$, $M=-100(x-16)$.

(2) 35, 200 lbs, at second support.

(5) At points distant 2·55 and 5·65 ft. from A.

(8) If x = distance from C then when

$x<\frac{a}{4}$, $M=-\frac{wx^2}{2\sqrt{2}}$ and when $x>\frac{a}{4}$, $M=\frac{a\sqrt{2}}{3}\left(x-\frac{a}{4}\right)-\frac{wx^2}{2\sqrt{2}}$,

where w = weight per unit length of rod.

(14) Tendency to break is same at a prop. and at the middle.

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